AN EASILY CHECKING CONDITION FOR THE STABILITY TEST OF A FAMILY OF POLYNOMIALS WITH NONLINEARLY PERTURBED COEFFICIENTS

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Abstracts In many cases of robust stability problems, the characteristic polynomial has real coefficients which are multilinear or nonlinear functions of uncertain parameters. For this set of polynomials, a new stability condition and an easily checking algorithm for reducing the conservatism of the stability bound are given. It is very useful for the new stability theorem to determine the stability region just in parameter space. Illustrative examples show that the presented method has larger stability bound in uncertain parameter space than others.

Keywords Hurwitz stable, A family of polynomials, Hermite-Biehler theorem, Sturm's theorem

1. INTRODUCTION

A family of transfer function including parametric uncertainty takes the following form.

$$G(s,q) = \frac{N(s,q)}{D(s,q)}\ ,\ q \in Q \tag{1}$$

$$Q := \{\, q \in \mathbb{R}^l \mid q_i^- \le q_i \le q_i^+ ,\, i=1,2,3,\cdots,l \,\} \tag{2}$$

where q is an uncertainty vector in hyperrectangle Q whose coefficients are dependent on q. $N(s,q)$ and $D(s,q)$ are polynomials in s. When such a open-loop system is followed by a linear controller, the closed-loop characteristic polynomial results in

$$\Delta(s,q) = \sum_{i=0}^{n} \delta_i(q) s^i\ ,\quad q \in Q \tag{3}$$

According to what dependency has the coefficients, eq.3 can be classified into three cases of uncertainty structure[1,2]; (i) interval polynomial family, (ii) polytopic polynomial family, (iii) multilinear/nonlinear uncertainty structure. In this note, we are concerned with the robust stability test of case (iii) of polynomials coefficients (case(iii)), whose coefficients are multilinear or nonlinear in uncertainty parameters. The classical Routh-Hurwitz test for case (iii) becomes quite impractical as long as the degree of polynomial is low. In some cases, it was shown that families of polynomials with complicated dependence on q can often be overbounded by interval polynomials family or polytopes of polynomials[1]. Then Kharitonov theorem[3] can be applied to check the Hurwitz stability. However the overbounding process gives conservative stability bounds for uncertain parameters which may or may not be tolerable. Pujara[4] has proposed a sufficient condition for reducing the conservatism of stability bounds above. This method is based on Hermite-Biehler theorem[5]. But it is not easy to apply Pujara's method to high degree of polynomials as well as polynomials including negative uncertainty parameters. Recently Anderson et al[6] presented sufficient conditions for the robust stability of only multilinear uncertainty structure. These conditions can be easily checked on the value of Jacobian determinants at certain corner points. In this paper, a new condition for stability of both multilinearly and nonlinearly uncertain polynomials is presented. It will be shown that the present method can reduce the conservatism considerably, comparing with those of overbounding method and Pujara's results. Here, we have introduced the monotonic argument property of Hurwitz polynomials and Sturm's theorem.

2. HURWITZ STABILITY OF POLYNOMIALS WITH NONLINEARLY PERTURBED COEFFICIENTS

Consider a real polynomial of continuous system

$$P(s) = a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n. \tag{4}$$

$P(s)$ can be written as

$$\begin{aligned} P(s) &:= P^{even}(s) + P^{odd}(s) \\ P(s) &:= P^{e}(s) + sP^{o}(s) \end{aligned} \tag{5}$$

where $P^{even}(s)$ and $P^{odd}(s)$ are the even and odd parts of $P(s)$ respectively.
Also,

$$P(j\omega):= P^{even}(\omega)+jP^{odd}(\omega)$$
$$P(j\omega):= P^{e}(\omega)+j\omega P^{o}(\omega) \tag{6}$$

Definition 2.1 (Interlacing or positive pair)[5]: A real polynomial $P(s)$ satisfies the interlacing property if (i) the leading coefficients of $P^{even}(s)$ and $P^{odd}(s)$ are of the same sign, and (ii) all the zeros of $P^{even}(s)$ and $P^{odd}(s)$ are distinct, lie on the imaginary axis, and alternate along it in the manners; for n=2m

$$0<\omega_{e,1}<\omega_{o,1}<\omega_{e,2}<\omega_{o,2}<\cdots<\omega_{e,m} \tag{7}$$

We present the following theorem

Theorem 2.2 (Hermite-Biehler Theorem)[5]: A real polynomial $P(s)$ is Hurwitz if and only if it satisfies the interlacing property. ■

Now, we are ready to state the main results of this paper

2.1 MAIN RESULTS

Define

$$Y^{o}(s):=\frac{P^{odd}(s)}{P^{even}(s)} \tag{8}$$

$$Y^{e}(s):=\frac{P^{o}(s)}{P^{e}(s)} \tag{9}$$

Theorem 2.3 : A real polynomial $P(s)$ satisfies the interlacing property if and only if either (i) or (ii) is satisfied

(i) $$\frac{dY^{o}(\omega)}{d\omega}>0, \quad \text{for } \omega\in[\,0,\ \infty) \tag{10}$$

(ii) $$\frac{dY^{e}(\omega)}{d\omega}>0, \quad \text{for } \omega\in(0,\ \infty) \tag{11}$$

Proof : (Necessity) Suppose that $P(s)$ has the interlacing property. Then it implies from Theorem 2.2 that $P(s)$ is Hurwitz. It is well known that if $P(s)$ is Hurwitz then all its coefficients are nonzero and have all positive sign. Suppose that n=2m. In this case $P^{e}(s)$ and $P^{o}(s)$ can be written as

$$P^{e}(s) = \prod_{i=1}^{m} a_{2m}(s^2+\omega_{e,i}^2),\ \omega_{e,i}>0 \tag{12}$$

$$P^{o}(s) = \prod_{j=1}^{m-2} a_{2m-1}(s^2+\omega_{o,j}^2),\ \omega_{o,j}>0 \tag{13}$$

From (8),(9),(10), and (11)

(14)

$$\begin{aligned} Y^{o}(s) &= \frac{P^{odd}(s)}{P^{even}(s)} \\ &= \frac{Ks(s^2+\omega_{o,1}^2)(s^2+\omega_{o,2}^2)\cdots}{(s^2+\omega_{e,1}^2)(s^2+\omega_{e,2}^2)\cdots} \\ &= Ks+\frac{2K_{11}s}{(s^2+\omega_{e,1}^2)}+\frac{2K_{12}s}{(s^2+\omega_{e,2}^2)}+\cdots \\ Y^{e}(s) &= \frac{P^{o}(s)}{P^{e}(s)} \\ &= \frac{K(s^2+\omega_{o,1}^2)(s^2+\omega_{o,2}^2)\cdots}{(s^2+\omega_{e,1}^2)(s^2+\omega_{e,2}^2)\cdots} \\ &= K+\frac{2K_{21}s}{(s^2+\omega_{e,1}^2)}+\frac{2K_{22}s}{(s^2+\omega_{e,2}^2)}+\cdots \end{aligned} \tag{15}$$

Since P(s) is Hurwitz by assumption, Theorem 2.2 states that all the zeros of $P^{e}(s), P^{o}(s), P^{even}(s)$ and $P^{odd}(s)$ lie on only $j\omega$ axis. Thus all the residue $K'_{ij}s$ $(i,j=1,2,\cdots,m)$ must be real and positive. Now it is easy to check that both eq.(10) and (11) satisfy.

(Sufficiency) To prove the sufficiency, we use the contradiction. Suppose that either (10) or (11) satisfies. If there is a set of solutions below as an example, without loss of generality,

$$0<\omega_{e,1}<\omega_{o,1}<\omega_{o,2}<\omega_{e,2}<\cdots \tag{16}$$

then necessarily both $Y^{e}(\omega)$ and $Y^{o}(\omega)$ would be increasing over $[0,\ \omega_{e,1}]$ and $[\omega_{e,1}, \omega_{o,1}]$ which be decreasing over$[\omega_{o,1}, \omega_{o,2}]$. It implies that $Y^{e}(\omega)$ and $Y^{o}(\omega)$ should have negative slope over $[\omega_{o,1}, \omega_{o,2}]$. This would contradict the assumption that they are monotonically increasing function with respect to ω. Thus the proof is completed. ■

Now we consider a family $P(s,q)$ of uncertain polynomial of the form (3).

$$\begin{aligned} P(s,q) &= \sum_{i=0}^{n} a_i(q)s^i,\ q_i\in[\,q_i^-,q_i^+] \\ &= a_n(q)s^n+a_{n-1}(q)s^{n-1}+\cdots+a_0(q) \end{aligned} \tag{17}$$

where we suppose that $a_i(q)$ depends on q either multilinearly or nonlinearly. Then a simple corollary of Theorem 2.3 is deriven.

Corollary 2.4 : A family of polynomials $P(s,q)$ is Hurwitz if and only if either

$$\frac{dY^{o}(\omega,q)}{d\omega}>0,\ \omega\in[\,0\infty]\ ,\forall q\in Q \tag{18}$$

or

$$\frac{dY^{e}(\omega,q)}{d\omega}>0\ ,\omega\in(0,\infty),\forall q\in Q \tag{19}$$

■

As the proof is very similar to that of Theorem 2.3, it is omitted.

At this stage, the problem is to find a less conservative method determining whether either the inequality (18) or (19) satisfies or not. Here we remark that (18)[or(19)] can be written as a family of real polynomials. Since the denominator of left side of (18)[or (19)] must be positive without regarding to ω, let its numerator be a real polynomials $f^o(x,q)$ and $f^e(x,q)$ with $x=\omega^2$ respectively. Then the condition (18)[or (19)] is equivalent to the problem checking whether the corresponding family of real polynomials $f^o(x,q)$ or ($f^e(x,q)$) has no real zeros for a given q $\in$Q. That is, (18) and (19) imply that

$$f^o(x,q) > 0,\ x\in[\ 0,\ \infty]\ ,q\in Q \tag{20}$$

$$f^e(x,q) > 0,\ x\in(0,\ \infty),q\in Q \tag{21}$$

The (20) and (21) can be evaluated by using the Sturm's Theorem[5, p.175-176]

A. The stability test of a family of nonlinear uncertain polynomials via Sturm's theorem.

Define

$$f_1(x,q):=f(x,q) \tag{22}$$

$$f_2(x,q):=\frac{d}{dx}f(x,q) \tag{23}$$

Then the Sturm's chain, $\{f_1(x,q), f_2(x,q),\cdots,f_m(x,q)\}$ can be always constructed easily via the Euclidean algorithm.

$$f_1(x,q)=\mu_1(x,q)f_2(x,q)-f_3(x,q)$$

$$\cdots\cdots\cdots$$

$$f_{k-1}(x,q)=\mu_{k-1}(x,q)f_k(x,q)-f_{k+1}(x,q) \tag{24}$$

$$\cdots\cdots\cdots$$

$$f_{m-1}(x,q)=\mu_{m-1}(x,q)f_m(x,q)$$

In other words, $-f_{k+1}(\cdot)$ is the remainder on dividing $f_{k-1}(\cdot)$ by $f_k(\cdot)$ and proceed until $f_m(\cdot)$ is a constant. Let $V(x,q)$ be the number of sign changes in the Sturm sequence. We state the following theorem without proof.

Theorem2.5 : A family of polynomials of the form (13) is Hurwitz if and only if

$$V(0,q)-V(\infty,q)=0,\quad \forall q\in Q \tag{25}$$

■

Since the Sturm chain is a sequence of real polynomials in x, $V(o,q)$ and $V(\infty,q)$ can be written in simpler constant form (see Appendix A). The Hurwitz stability bounds of $P(s,q)$ which are evaluated from (20) and (21) by means of Theorem 2.5 may be quite different. In most cases, if the degree of $P(s,q)$ is even, the condition (20) gives bigger stability bound than that of (21), but if $P(s,q)$ has odd degree, then the condition (21) is better stability bound in uncertain parameter space. Here we make a point that the stability condition via Theorem 2.5 can be determined in only coefficient space without evaluation of roots of the given polynomial family.An illustrative example is given below.

Example 2.1 [1]: Consider the family of polynomials

$$\begin{aligned}p(s,q)= {} & s^4+(q_1+q_2+2.56)s^3+(q_1q_2+2.06q_1+1.561q_2\\ & +2.871)s^2+(1.06q_1q_2+4.841q_1+1.561q_2+3.164)s\\ & +(4.032q_1q_2+3.773q_1+1.985q_2+1.853)\\ = {} & a_4s^4+a_3s^3+a_2s^2+a_1s+a_0\end{aligned} \tag{26}$$

We want to find (a) the stability region of uncertain parameter vector $q^T=[\ q_1\ q_2]$ ranged over $q_i\in[\ -5,5]$, and (b) to determine the maximum bound of q$\in l^\infty$ so that can be guaranteed the stability subject to $q_i\geq 0$ (i=1,2). Substituting (20) into (A-3)~(A-6) in Appendix A and evaluating them numerically, we get the solution of (a) shown in Fig.2.1 and l^∞ stability bound of q over positive range are given as $q_1\in[\ 0,\ 0.272\ ]$ and $q_2\in[\ 0,\ 0.544\ ]$. All the results are obtained from just coefficient space without evaluation of roots of the given polynomial family.

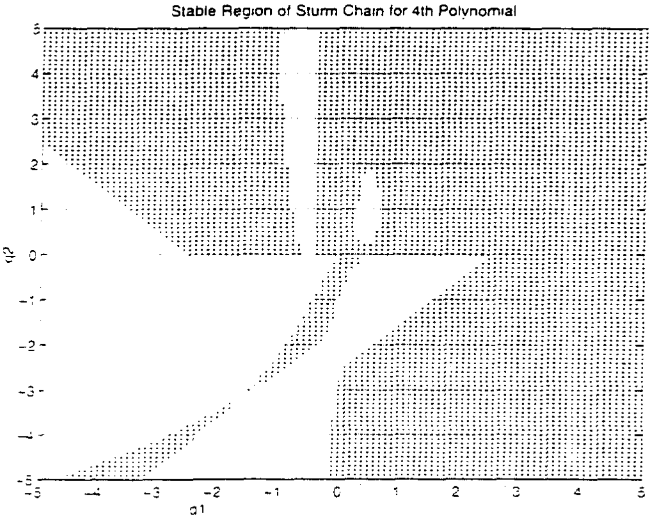

Fig.2.1 The Stability region(shaded) of uncertain parameter vector q over $q_i\in[\ -5,5]$ (i=1,2) for Example2.1.

B. An easily checking algorithm via Descartes' rule of sign.

In previous section A, we present a compact algorithm for Hurwitz stability test of a family of nonlinearly perturbed polynomials but it is some or less conservative in view of computation. Now this section discusses an easily checking algorithm which can be used effectively in some cases. Corollary 2.4 states that either (20) or (21) is necessary and sufficient condition for stability of $P(s,q)$ in (17). The condition (20)[or(21)] implies that a family of real polynomials $f^o(x,q)$ has no real distinct roots for a given q∈Q over $x \in [0, \infty]$. There is an interesting result named Descartes' rule of sign for the purpose of checking such a problem.

Theorem2.6 (Descartes' rule of signs)[7]: If $P(x)$ is a polynomial with real coefficients, then the number of positive real solutions of $P(x)=0$ either is equal to the number of variations in sign occurring in the coefficients of $P(x)$, or else is less than this number by an even natural number ■

First, calculating both upper and lower bounds of each coefficients in (20) and (21) with respect to q∈Q independently, we can determine variation in sign as a sufficient condition. By means of Theorem 2.6, we conclude that there is at least a real root if the variation in sign occurs in odd numbers. If the number is even, we can regard that it has at least two real roots even if nothing is true. To show the usefulness of this method, two examples in literatures[1,4] are considered and compared with their results.

Example2.2[1,p.242]: Consider a family of polynomials.

$$P(s,q) = s^3 + (3q_1^3 + q_1^2 q_2 + q_1 q_2 + 3q_1 + 10)s^2 + (4q_1^2 + q_2^2 + 15)s + (6q_1 q_2 + 17) \quad (27)$$

From (18)-(21), we obtain

$$f^o(x,q) = a_2(q)a_3(q)x^2 + [\, a_1(q)a_2(q) - 3a_0(q)a_3(q)\,]\, x + a_1(q)a_0(q) \quad (28)$$

$$f^e(x,q) = a_1(q)a_2(q) - a_3(q)a_0(q) \quad (29)$$

Evaluating the sufficient bounds of q∈Q near $q^T = [\,0, 0\,]$ such that variation in sign in (28) and (29) does not occur, the results are in Table 2.1

Table 2.1 Sufficient bound of stability for example 2.2

Method	q_1	q_2
Kharitonov Overbounding Method	[-1.11935 1.11935]	[-2.2387 2.2387]
eq. (28)	[-1.0148 1.0148]	[-2.0296 2.0296]
eq. (29)	[-1.165 1.165]	[-2.325 2.325]

Example 2.3 [4]: Find a positive stability bound ε of a family of polynomials with uncertainty

$$P(s,q) = 2s^3 + (2q_1 + 2q_2 + q_3^2)s^2 + (2q_1 q_2 + 2q_3 + 5)s + (2q_1^2 + 2q_2^2 + 3q_3^2) \quad (30)$$

with uncertainty $1-\varepsilon < q_i < 1+\varepsilon \quad (i=1,2,3)$

Compare with (a) Kharitonov overbound method, and (b) Pujara's method, the proposed method gives much larger uncertainty bound for stability than other ones. The results are in Table 2.2

Table 2.2 Sufficient bound of stability for example 2.3

Method	q_i (i=1,2,3)
Kharitonov Overbounding Method	[1.08 1.92]
Pujara's Method	[0.77 2.23]
The Proposed Method using Descartes' rule	[0.09 2.91]

3. CONCLUSION

In this paper, a robust Hurwitz stability of a family of polynomials with complicated dependence on uncertain parameter is presented. New stability condition transforms the test problem in the complex function space into the problem checking whether the deriven family of real polynomials have any roots on the positive real axis.

By means of Sturm's theorem, this condition can be very useful to determine the stability region in a given uncertain parameter space. Also, we propose an easily checking algorithm reducing the conservatism of evaluation via Descartes' rule of sign. Some numerical examples with multilinear and nonlinear uncertainty structure are given to illustrate the given method.

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Appendix A :

Constant forms of $V(0,q)$ and $V(\infty,q)$

For simplicity, we consider the 4th degree of $p(s,q)$ of the form (17).

$$p(s,q) = a_4(q)s^4 + a_3(q)s^3 + a_2(q)s^2 + a_1(q)s + a_0(q), \quad q \in Q \qquad \text{(A-1)}$$

where assume that $a_i(q)$ depends on $q \in Q$ either multilinearly or nonlinearly. Eq(18) give rise to the following

$$f_0(x,q) = b_3(q)x^3 + b_2(q)x^2 + b_1(q)x + b_0(q) \qquad \text{(A-2)}$$

where

$$\begin{aligned} b_3(q) &= a_3(q)a_4(q) \\ b_2(q) &= a_2(q)\, a_3(q) - 3a_1(q)\, a_4(q) \\ b_1(q) &= a_1(q)\, a_2(q) - 3a_0(q)\, a_3(q) \\ b_0(q) &= a_0(q)\, a_1(q) \end{aligned} \qquad \text{(A-3)}$$

Let $f_1^0(x,q) = f^0(x,q)$.

Appling (24) to (A-2), we get the Sturm sequence

$$\begin{aligned} f_2^0(x,q) &= \frac{d}{dx} f^0(x,q) \\ &= 3b_3(q)x^2 + 2b_2(q)x^2 + b_1(q) \\ f_3^0(x,q) &= \left(\frac{2b_2^2(q)}{9b_3(q)} - \frac{2b_1(q)}{3}\right)x + \\ &\quad \left(\frac{b_1(q)\, b_2(q)}{9b_3(q)} - b_0(q)\right) \\ &= c_1(q)x + c_0(q) \\ f_4^0(x,q) &= \frac{2c_0(q)b_2(q)}{c_1(q)} - 3\left(\frac{c_0(q)}{c_1(q)}\right)^2 b_3(q) - b_1(q) \\ &= d_0(q) \end{aligned} \qquad \text{(A-4)}$$

Thus

$$\begin{aligned} V(0,q) &= [\ f_1^0(0,q)\ f_2^0(0,q)\ f_3^0(0,q)\ f_4^0(0,q)] \\ &= [\ b_0(q)\ b_1(q)\ c_0(q)\ d_0(q)]\ , \quad q \in Q \end{aligned} \qquad \text{(A-5)}$$

$$\begin{aligned} V(\infty,q) &= [\ f_1^0(\infty,q)\ f_2^0(\infty,q)\ f_3^0(\infty,q)\ f_4^0(\infty,q)] \\ &= [\ b_3(q)\ b_3(q)\ c_1(q)\ d_0(q)]\ , \quad q \in Q \end{aligned} \qquad \text{(A-6)}$$

$V(0,q)$ and $V(\infty,q)$ corresponding to (21) can be developed similarly.