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OPTIMIZATION OF ERROR PATH MODEL IN FILTERED-X LMS ALGORITHM FOR NARROW BAND NOISE SUPPRESSION

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Abstracts Adaptive algorithms based on gradient adaptation have been extensively investigated and successfully joined with active noise/vibration control applications. The Filtered-X LMS algorithm became one of the basic feedforward algorithms in such applications, but still is not fully understood. The error path model effect on the Filtered-X LMS algorithm has been under the investigation and some useful properties related stability has been discovered. We are interested in utilizing the fact that the model error caused by the difference of the true error path and its model does not always related to the performance degradation. In this paper, we presents the way optimizing the error path model in a view point of convergence speed of Filtered-X LMS algorithm for pure tone noise suppression application without any performance loss at steady state.

Keywords Filtered-X LMS, Active noise control, Narrow band noise, Optimization

1. INTRODUCTION

Adaptive filter updated by its gradient of cost function is widely used in active noise and vibration control applications. Among them, the Filtered-X LMS algorithm[1,2,3] that can cope with the systems having an *error path* - an auxiliary path between the control speaker input and the error microphone output - has been a popular algorithm for its low computational burden and easy programming.

In ANC (Active Noise Control) system the reference signal measured from noise source is an excitation signal. It is filtered through adaptive weight whose output is a cancellation signal to the secondary speaker. The weights of adaptive filter are updated to the direction of minimizing the instantaneous squared error measured from the error microphone. The update process requires prefiltering of the reference signal through the model of error path to ensure convergence of the cost function - instantaneous squared error. Allowable model error is confined only by phase error between the actual error path and its model, and its limit is ±90°.[2,3,4]

If the degree of freedom of reference signal does not exceed that of the adaptive filter, i.e. *exact or underdetermined case*, the steady state weight values are the same as the optimum one even when the model error presents while phase error does not exceed ±90°.[4] In this case, without any steady state performance loss, we have another design parameter - *error path model* - whose constraint is given by making the algorithm converge. The typical situation of the exact or underdetermined case can be found in narrow band noise suppression application where the number of adaptive filter weight is not less than two which is the degree of freedom of pure tone reference signal.

Here, we present the error path model design procedure by introducing an *intentional optimal phase error* to achieve fast convergence of Filtered-X LMS algorithm.

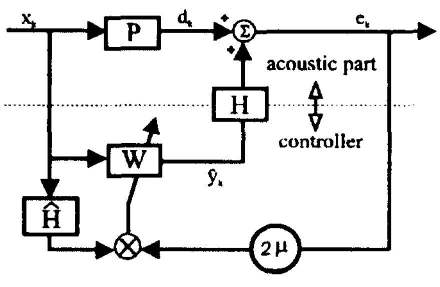


Fig.1 Block diagram of ANC system.

2. GRADIENT DESCRIPTION

The system we are concern about is the ANC system as shown in Fig.1. This control system, having an adaptive weight vector W, is excited with pure tone single reference input x_k of ω frequency and its output $\hat{y}_k$ is intended to cancel the primary noise at the error microphone location. P represents the plant from the reference to the error microphone, H and $\hat{H}$ represent the error path and its model, respectively and μ is an update gain.

In Filtered-X LMS algorithm, cost function J_k is defined as instantaneous squared error as follows :

$$J_k = e_k^2 = \bar{e}_k(\omega)e_k(\omega) \ , \tag{1}$$

where $e_k(\omega)$ is Fourier transform of error signal e_k at the excitation frequency ω, and superscript $^-$ represents complex conjugate. Assuming slow change of weight values, from diagram of Fig.1 we can express $e_k(\omega)$ as

$$e_k(\omega) = \{P(j\omega) + H(j\omega)W_k(j\omega)\}\ x_k(\omega) \ , \tag{2}$$

where $P(j\omega)$, $H(j\omega)$ and $W_k(j\omega)$ are complex transfer functions at frequency ω of plant, error path, and adaptive filter, respectively and $x_k(\omega)$ is Fourier transform of reference signal x_k. The error represented in (2) is wrong in strict sense, because the dynamics of error path makes e_k depend on the past weights W_{k-1}, W_{k-2}, ⋯ as well as W_k. However the exact formulation representing such effect needs very complicated mathematics, so we assume slow change of weight values for simplicity.

Complex weight value at ω, $W_k(j\omega)$, can be expressed by conventional FIR type W_k of length L+1 and time to frequency domain transformation vector $Z_L(j\omega)$ of length L+1 as follows :

$$W_k(j\omega) = Z_L^t(j\omega)W_k \ , \tag{3}$$

$$Z_L(j\omega) = \frac{1}{\sqrt{L+1}}\left[1 \ \ e^{-j\omega T} \ \ e^{-j2\omega T} \ \cdots \ e^{-jL\omega T}\right]^t \ , \tag{4}$$

where T is sampling interval.

Using (2) and (3), differentiation of cost function J_k with respect to W_k gives true instantaneous gradient vector ∇W_k in time domain as

$$\nabla W_k = 2\,\mathrm{Re}\{\bar{e}_k(\omega)H(j\omega)Z_L(j\omega)x(\omega)\} \ . \tag{5}$$

Since we should use estimated error path model $\hat{H}$ instead of real error path H in gradient calculation, estimated gradient is

$$\hat{\nabla} W_k = 2\,\mathrm{Re}\left\{\bar{e}_k(\omega)\hat{H}(j\omega)Z_L(j\omega)x(\omega)\right\} . \tag{6}$$

With convergence parameter μ and estimated gradient vector, we can get the update form of Filtered-X LMS:

$$W_{k+1} = W_k - \mu\hat{\nabla} W_k \ . \tag{7}$$

For further investigation, let's express the prescribed transfer functions in terms of real and imaginary part as follows

$$Z_L(j\omega) = Z_{Lx} + jZ_{Ly} \ , \tag{8.a}$$

$$P(j\omega) = p\cos\phi_p + jp\sin\phi_p \ , \tag{8.b}$$

$$\hat{H}(j\omega) = \hat{h}\cos\phi_{\hat{h}} + j\hat{h}\sin\phi_{\hat{h}} \ , \tag{8.c}$$

$$H(j\omega) = h\cos\phi_h + jh\sin\phi_h \ , \tag{8.d}$$

where p, $\hat{h}$, and h are magnitudes and ϕ_p, $\phi_{\hat{h}}$, and ϕ_h are phases of each transfer function. Then, using (2) and (6), estimated gradient in (7) can be rewritten in matrix form

$$\hat{\nabla} W_k = 2\{\mathbf{S}_L\mathbf{H}^t\hat{\mathbf{H}}\mathbf{S}_L^t\}W_k + 2\{\mathbf{S}_L\hat{\mathbf{H}}\mathbf{P}\} \ , \tag{9}$$

where matrix $\mathbf{S}_L$, $\mathbf{H}$, $\hat{\mathbf{H}}$, and $\mathbf{P}$ are defined as follows

$$\mathbf{S}_L = \begin{bmatrix} Z_{Lx} & Z_{Ly} \end{bmatrix},$$

$$\mathbf{H} = qh\begin{bmatrix} \cos\phi_h & \sin\phi_h \\ -\sin\phi_h & \cos\phi_h \end{bmatrix}, \quad \hat{\mathbf{H}} = q\hat{h}\begin{bmatrix} \cos\phi_{\hat{h}} & \sin\phi_{\hat{h}} \\ -\sin\phi_{\hat{h}} & \cos\phi_{\hat{h}} \end{bmatrix}, \tag{10}$$

$$\mathbf{P} = qp\begin{bmatrix} \cos\phi_p \\ \sin\phi_p \end{bmatrix} \quad , \ q = |x(\omega)|$$

Note that 2 by 2 matrices $\mathbf{H}$ and $\hat{\mathbf{H}}$ have the form of rotation matrix. The coefficient matrix $\mathbf{S}_L\mathbf{H}^t\hat{\mathbf{H}}\mathbf{S}_L^t$ at the left side of W_k in (9) is *cross correlation matrix* of the two filtered reference signals through H and $\hat{H}$. This matrix is very important in convergence process. Let's introduce a matrix $\mathbf{K}$ representing the cross correlation matrix for easy manipulation of further analysis

$$\mathbf{K} = \mathbf{S}_L\mathbf{H}^t\hat{\mathbf{H}}\mathbf{S}_L^t \quad \text{; cross correlation matrix.} \tag{11}$$

3. DECOMPOSITION OF K

Let's introduce a parameter θ representing the phase difference between $H(j\omega)$ and $\hat{H}(j\omega)$, then $\mathbf{H}^t\hat{\mathbf{H}}$ term in matrix $\mathbf{K}$ is rearranged in terms of θ as follows :

$$\theta = \phi_{\hat{h}} - \phi_h \ , \tag{12}$$

$$\mathbf{H}^t\hat{\mathbf{H}} = q^2h\hat{h}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} . \tag{13}$$

Note that $\mathbf{H}^t\hat{\mathbf{H}}$ contains rotation matrix of angle θ which causes unsymmetricity of cross correlation matrix. If θ=0, $\mathbf{K}$ is proportional to auto correlation matrix of filtered reference signal.

Introducing a L+1 by 2 matrix $\mathbf{U}_L$ whose column space of $\mathbf{S}_L$ satisfying $\mathbf{U}_L^t\mathbf{U}_L = \mathbf{I}$, a 2 by 2 diagonal matrix Λ having eigenvalues of $\mathbf{K}$ on its diagonal part, and a 2 by 2 invertable matrix $\mathbf{Q}$, $\mathbf{K}$ is decomposed into the following form (see Appendix for details)

$$\mathbf{K} = \mathbf{U}_L\mathbf{Q}\Lambda\mathbf{Q}^{-1}\mathbf{U}_L^t \tag{14}$$

where

$$\mathbf{U}_L = \mathbf{C}_L\Sigma_L^{-1} \tag{15.a}$$

$$\mathbf{Q} = \left(\alpha_L\cos\theta + \sqrt{1-\alpha_L^2}\sin\theta\right)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \beta_L\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{15.b}$$

$$\Lambda = \mathrm{diag}[\lambda_1, \lambda_2] = \tfrac{1}{2}q^2h\hat{h}\cos\theta\begin{bmatrix} 1+\dfrac{\beta_L}{|\cos\theta|} & 0 \\ 0 & 1-\dfrac{\beta_L}{|\cos\theta|} \end{bmatrix} \tag{15.c}$$

where $\mathbf{C}_L$, Σ_L, α_L, and β_L are defined as follows :

$$\mathbf{C}_L = \frac{1}{\sqrt{L+1}}\begin{bmatrix} \cos\frac{L}{2}\omega T & \sin\frac{L}{2}\omega T \\ \cos(\frac{L}{2}-1)\omega T & \sin(\frac{L}{2}-1)\omega T \\ \vdots & \vdots \\ \cos(\frac{L}{2}-1)\omega T & -\sin(\frac{L}{2}-1)\omega T \\ \cos\frac{L}{2}\omega T & -\sin\frac{L}{2}\omega T \end{bmatrix} \tag{16.a}$$

$$\Sigma_L = \begin{bmatrix} \sqrt{\dfrac{1+\alpha_L}{2}} & 0 \\ 0 & \sqrt{\dfrac{1-\alpha_L}{2}} \end{bmatrix} \tag{16.b}$$

$$\alpha_L = \frac{\sin(L+1)\omega T}{(L+1)\sin\omega T} \tag{16.c}$$

$$\beta_L = \sqrt{\alpha_L^2 - \sin^2\theta} \tag{16.d}$$

The variable α_L is a function of filter length L+1 and ωT which is 2π times of normalized target frequency f/f_s, i.e. $2\pi f/f_s$.

Using the decomposed form of $\mathbf{K}$, update equation (7) can be rewritten in modal domain weight vector $\mathbf{Q}^{-1}\mathbf{U}_L^t W$ as follows

$$(\mathbf{Q}^{-1}\mathbf{U}_L^t W_{k+1}) = [1-2\mu\Lambda](\mathbf{Q}^{-1}\mathbf{U}_L^t W_k) - 2\mu\mathbf{Q}^{-1}\mathbf{U}_L^t\mathbf{S}_L\hat{\mathbf{H}}\mathbf{P} \ . \tag{17}$$

Clearly, the convergence characteristic depends on eigenvalues in Λ, though the eigenvectors - the columns of $\mathbf{U}_L\mathbf{Q}$ - may not orthogonal for non-zero value of θ. From (15.c), real part of the two eigenvalues become negative when $|\theta|$ exceeds $90°$ which implies the algorithm is unstable for any positive μ.

4. EIGENVALUE SPREAD

The convergence speed is of course a function of the convergence parameter μ, but maximum attainable speed is limited by the eigenvalue spread of $\mathbf{K}$. So from now on, when we use the terminology 'convergence speed', we are mainly concern about the 'eigenvalue spread'.

At most, only two eigenvalues of $\mathbf{K}$ are non-zero for narrow band reference input. In this section, we are going to represent the spread of the two eigenvalues. By investigating the parameters included in eigenvalue spread, we could get a condition that makes it unity to get fast convergence speed for any initial guess of weight.

From (15.c) the eigenvalue spread r is

$$r=\frac{|\cos\theta|-\beta_L}{|\cos\theta|+\beta_L}\ . \tag{18}$$

The resulting eigenvalue spread depends on three parameters. First one is model phase error θ. Second one is adaptive filter length L+1, and the last one is f/f_s.

Eigenvalue spread r becomes unity when $\beta_L = 0$, which is identical to

$$\sin^2\theta=\alpha_L^2\ . \tag{19}$$

Using the value α_L defined in (16.c), optimum model error θ_{opt} for fast convergence is determined as follows

$$\theta_{opt}=\pm\sin^{-1}\left[\frac{\sin(L+1)\omega T}{(L+1)\sin\omega T}\right]\ . \tag{20}$$

Note that there are two values of θ_{opt} whose absolute values are equal to each other. The case of $\theta_{opt} = 0$ means there needs no extra effort to increase convergence speed provided that exact phase model at target frequency is obtained.

One condition that makes θ_{opt} zero occurs when $\sin(L+1)\omega T$ is zero while $\sin\omega T$ is not zero, i.e. normalized target frequency f/f_s is equal to $k/2(L+1)$ for k = 1, 2, ⋯, L. If we could set f/f_s to one of these values by adjusting f_s, no modification of exact error path model is necessary, while in case that adjustment of f_s is not allowed, introduction of optimal phase error θ_{opt} could give faster convergence. Another condition that makes θ_{opt} close to zero is increasing L, which requires more computation.

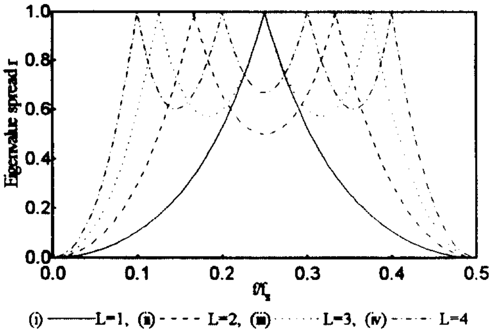


Fig.2 Eigenvalue spread along f/f_s with θ = 0.

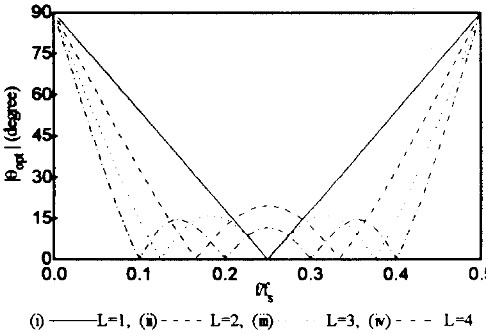


Fig.3 $|\theta_{opt}|$ along f/f_s.

Fig.2 shows eigenvalue spread when θ = 0 along the normalized target frequency f/f_s using (18) with L = 1, 2, 3, 4. Fig.3 shows $|\theta_{opt}|$ along f/f_s using (20) with L = 1, 2, 3, 4. From Fig.3,4, only L points at $f/f_s = k/2(L+1)$ for k = 1, 2, ⋯, L satisfy r = 1. Elsewhere introduction of $|\theta_{opt}|$ is needed to make r unity.

5. OPTIMUM MODEL DESIGN

We will now describe how to modify error path model to get fast convergence. We assume that error path model $\hat{H}$ is FIR type with M+1 taps and is estimated prior to update of Filtered-X LMS which can be said to be 'exact' around the known target frequency ω. This model can be obtained by exciting the error path with signal containing periodic signal of frequency ω.

$$\hat{H}=\begin{bmatrix}\hat{h}_0 & \hat{h}_1 & \cdots & \hat{h}_M\end{bmatrix}^t \text{ ; close to H around } \omega \text{ frequency.} \tag{21}$$

We obtain a transformation matrix **F** through which optimal error path $\hat{H}_{opt}$ is given by $\mathbf{F}\hat{H}$. The procedure getting **F** is described below: Decomposing $\hat{H}(j\omega)$ to real and imaginary parts using column vector $Z_M(j\omega)$ of length M+1, we can build a column vector $\begin{bmatrix}Z_{Mx}^t\hat{H} & Z_{My}^t\hat{H}\end{bmatrix}^t$, and rotate it with angle of θ_{opt}. The rotated vector must be the same as $\begin{bmatrix}Z_{Mx}^t\hat{H}_{opt} & Z_{My}^t\hat{H}_{opt}\end{bmatrix}^t$. Applying pseudo inverse of $\mathbf{S}_M^t$ to the obtained equality, we get transformation matrix **F** as follows.

$$\hat{H}_{opt}=\mathbf{F}\hat{H}\ , \tag{22.a}$$

$$\begin{aligned}\mathbf{F}&=(\mathbf{S}_M^t)^+\begin{bmatrix}\cos\theta_{opt} & -\sin\theta_{opt}\\ \sin\theta_{opt} & \cos\theta_{opt}\end{bmatrix}\mathbf{S}_M^t\\ &=\mathbf{C}_M\Sigma_M^{-2}\begin{bmatrix}\cos\theta_{opt} & -\sin\theta_{opt}\\ \sin\theta_{opt} & \cos\theta_{opt}\end{bmatrix}\mathbf{C}_M^t\end{aligned}\ , \tag{22.b}$$

where superscript $^+$ is pseudo inverse, and two matrices $\mathbf{C}_M$ and Σ_M are in (16.a) and (16.b), respectively.

Overall procedure of getting $\hat{H}_{opt}$ is thus as follows.

- Obtain $\hat{H}$ by modeling the error path.
- Select adaptive filter length L+1 and ωT.
- Calculate θ_{opt} from (20).
- Calculate $\hat{H}_{opt}$ from (22).

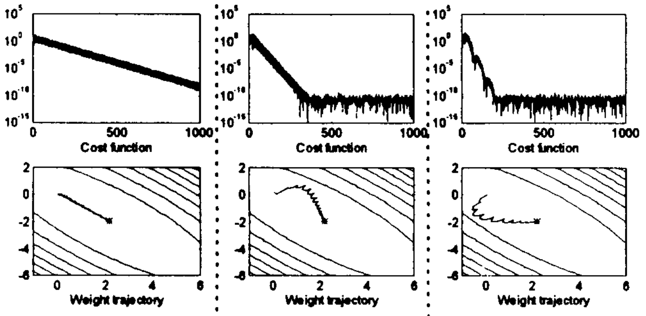
6. COMPUTER SIMULATIONS

In computer simulation, adaptive weight length is set to 2, i.e., $W(z)=w_0+w_1z^{-1}$, and the plant P(z) and error path H(z) are chosen as follows

$$P(z)=0.3z^{-6}+z^{-7}+2z^{-8}+z^{-9}+0.1z^{-10}\ ,$$
$$H(z)=z^{-4}+2z^{-5}\quad (\text{ i.e. } H=[\,0\ 0\ 0\ 0\ 1\ 2]^t\ ).$$

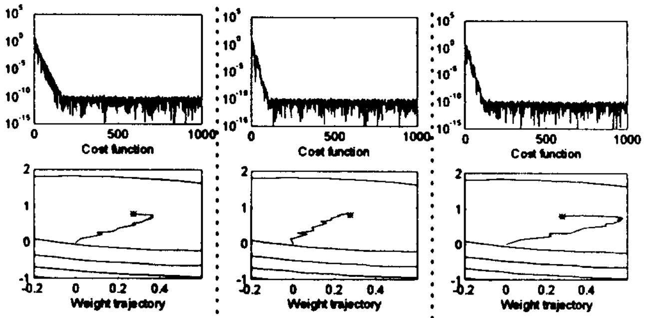
We chose 30% of maximum convergence parameter μ for all cases and assume there is white measurement noise in e_k whose variance is $10^{-12}/3$. Fig.4 shows time domain squared error e_k^2 and weight trajectory when $f/f_s = 0.2$ for (a) conventional $\hat{H}$, (b) $\hat{H}_{opt}$ using $|\theta_{opt}|$, (c) $\hat{H}_{opt}$ using $-|\theta_{opt}|$. Fig.5 and Fig.6 show the same when $f/f_s = 0.4$ and $f/f_s = 0.8$, respectively.

Computer simulations including proposed optimum error path model $\hat{H}_{opt}$ using $|\theta_{opt}|$ or $-|\theta_{opt}|$ leads fast convergence of cost function compared to the conventional exact modeling. The optimum error path speed up the convergence speed more for smaller r of conventional $\hat{H}$. The trajectory path is independent of the convergence speed, but is related with the right eigenvectors of $\mathbf{K}$ which are columns of $\mathbf{U}_L\mathbf{Q}$.



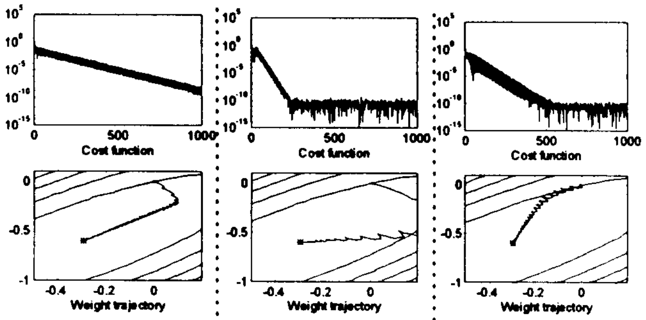
(a) conventional $\hat{H}$	(b) $\hat{H}_{opt}$ using $\lvert\theta_{opt}\rvert$	(c) $\hat{H}_{opt}$ using $-\lvert\theta_{opt}\rvert$
r = 0.1056	r = 1	r = 1
μ= 0.0061	μ= 0.0105	μ= 0.0111

Fig.4 Cost function and weight trajectory (w_0, w_1) when $f/f_s = 0.2$ ($|\theta_{opt}| = 54°$). (* : optimum weight)



(a) conventional $\hat{H}$	(b) $\hat{H}_{opt}$ using $\lvert\theta_{opt}\rvert$	(c) $\hat{H}_{opt}$ using $-\lvert\theta_{opt}\rvert$
r = 0.5279	r = 1	r = 1
μ= 0.0109	μ= 0.0124	μ= 0.0138

Fig.5 Cost function and weight trajectory (w_0, w_1) when $f/f_s = 0.4$ ($|\theta_{opt}| = 18°$). (* : optimum weight)



(a) conventional $\hat{H}$	(b) $\hat{H}_{opt}$ using $\lvert\theta_{opt}\rvert$	(c) $\hat{H}_{opt}$ using $-\lvert\theta_{opt}\rvert$
r = 0.1056	r = 1	r = 1
μ= 0.0230	μ= 0.0490	μ= 0.0263

Fig.6 Cost function and weight trajectory (w_0, w_1) when $f/f_s = 0.8$ ($|\theta_{opt}| = 54°$). (* : optimum weight)

7. CONCLUSIONS

An optimal transformation of error path model that makes the Filtered-X LMS algorithm converge with fast speed for narrow band noise suppression application is presented. This is done by inserting intentional phase error to FIR type error path model which is well estimated around the target frequency. For even with short adaptive filter length, we could achieve fast convergence speed by applying the proposed method. Obtained convergence characteristic is similar to LS method that uses inversion of correlation matrix because the eigenvalue spread is adjusted to unity. The merit of proposed method compared to LS method is that there is no increase in computational load in real time application because modifying the model can be done prior to control stage.

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APPENDIX : PROOF OF $\mathbf{S}_L\mathbf{H}^t\hat{\mathbf{H}}\mathbf{S}_L^t = \mathbf{U}_L\mathbf{Q}\Lambda\mathbf{Q}^{-1}\mathbf{U}_L^t$

Let's consider singular value decomposition of $\mathbf{S}_L$

$$\mathbf{S}_L = \mathbf{U}_L\Sigma_L\mathbf{V}_L^t \ , \qquad \text{(A.1)}$$

where L+1 by 2 matrix $\mathbf{U}_L$ and 2 by 2 matrix Σ_L are given in (15.a) and (16.b), while $\mathbf{V}_L$ is 2 by 2 rotation matrix as follows

$$\mathbf{V}_L = \begin{bmatrix} \cos\frac{1}{2}\omega T & \sin\frac{1}{2}\omega T \\ -\sin\frac{1}{2}\omega T & \cos\frac{1}{2}\omega T \end{bmatrix} . \qquad \text{(A.2)}$$

Using the property that $\mathbf{V}_L$, $\mathbf{H}$ and $\hat{\mathbf{H}}$ are scaled rotation matrices and thus commutable to each other, the cross correlation matrix $\mathbf{K}$ can be rearranged as

$$\begin{aligned} \mathbf{K} &= \mathbf{S}_L\mathbf{H}^t\hat{\mathbf{H}}\mathbf{S}_L^t \\ &= \mathbf{U}_L\Sigma_L\mathbf{V}_L^t\mathbf{H}^t\hat{\mathbf{H}}\mathbf{V}_L\Sigma_L\mathbf{U}_L^t \ . \qquad \text{(A.3)} \\ &= \mathbf{U}_L\Sigma_L\mathbf{H}^t\hat{\mathbf{H}}\Sigma_L\mathbf{U}_L^t \end{aligned}$$

There exist an invertable matrix $\mathbf{Q}$ and diagonal matrix Λ such that

$$\Sigma_L\mathbf{H}^t\hat{\mathbf{H}}\Sigma_L = \mathbf{Q}\Lambda\mathbf{Q}^{-1} \ , \qquad \text{(A.4)}$$

where precise expression of $\mathbf{Q}$ and Λ are given in (15.b) and (15.c), respectively. Thus, combining (A.3) and (A.4) with variables as in (15) and (16), $\mathbf{K}$ is decomposed into $\mathbf{U}_L\mathbf{Q}\Lambda\mathbf{Q}^{-1}\mathbf{U}_L^t$.