NOTE ON BEHAVIOR OF A COUPLED NONAUTONOMOUS ORDINARY DIFFERENTIAL EQUATION

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Abstract Stability of a coupled nonautonomous ordinary differential equation is investigated. Asymptotic convergence to zero of a part of state vector is additionally shown, otherwise only uniform stability could have been concluded by the Lyapunov direct method. Obtained results could be particularly useful in analysis of nonautonomous systems in which the invariance principle does not hold. An illustrating example is given.

Keywords Nonautonomous system, Lyapunov Stability, Convergence, Adaptive Control

1. INTRODUCTION

It is well known that for a linear nonautonomous system such that $\dot{x}(t) = A(t)x(t), x(0) = x_0$, where $x(\cdot) \in R^n, t \geq 0$, the eigenvalue analysis is not sufficient for determining the stability [1] [8], i.e. the fact that the real parts of all the eigenvalues of $A(t)$ at every time instant are bounded above by $-\delta, \delta > 0$, does not imply the stability of the time-varying system. A pioneering example of Markus and Yamabe [10] (see also [8] p.184 Example 109) actually shows that the solution grows without bound as $t \to \infty$ even if the spectrums of $A(t)$ remain at fixed locations in the open left-half plane for all $t \geq 0$. Several recent results for the exponential stability of linear systems including infinite dimensional systems can be found in [2][3].

For a nonlinear system two approaches are usually taken in determining stability. First one is the Lyapunov's direct method which analyzes qualitatively the behavior of dynamic system by utilizing the Lyapunov function (resembling total energy in some sense). And second one is the Lyapunov's indirect method which enables one to draw conclusions about a nonlinear system by studying the behavior of a linear system obtained through linearization. Furthermore if the given system is nonautonomous, assertion such as asymptotic stability would be more demanding since it requires that the derivative of Lyapunov function, $-\dot{V}(t)$, is (locally) positive definite. However for special cases like autonomous or periodic system the invariance principle (LaSalle's theorem) is known to hold [8, p.156], therefore it is possible to conclude asymptotic stability even in cases where $-\dot{V}(t)$ is not locally positive definite

In this note the asymptotic convergence to zero of a part of solution of a coupled nonautonomous system is investigated. The coupled dynamic system is assumed to permit a Lyapunov function and the time derivative of the Lyapunov function is negative semidefinite involving only part of the state. Therefore uniform stability can only be concluded from the Lyapunov direct method [8, p.148, Theorem 9]. Note also that the invariance principle does not hold for general nonautonomous system. In this note however the asymptotic convergence to zero of the partial state of the coupled system will be shown with additional assumptions, which are not restrictive, which appear in the derivative of the Lyapunov function. The contribution of the note is to show additionally the asymptotic convergence to zero of first part of state vector of coupled system (1)-(2) below, otherwise only uniform stability could have been concluded by the Lyapunov method. An illustrating example for the obtained results will be given.

Definition: $z(t)$ is said to be a solution of the differential equation

$$\dot{z} = f(t,z), z(\tau) = \bar{z}$$

where $z: I = [\tau, \tau + a] \to R^n$, $f: D \equiv I \times B \to R^n$ where $B = \{z \in R^n : \|z - \bar{z}\| \leq b\}$, if $(t, z(t)) \in D$ for $t \in I$, and $z(t)$ satisfies the given differential equation for all $t \in I$.

2. MAIN RESULTS

Theorem: Consider a coupled nonautonomous system as

$$\dot{x} = f(t,x,y),\ x(0) = x_o \quad (1)$$

$$\dot{y} = g(t,x,y),\ y(0) = y_o \quad (2)$$

where $x \in R^n, y \in R^m$. The following assumptions are made.

(i) $f(t,0,0) = 0$. $g(t,0,0) = 0$. f and g are piecewise continuous in t, and continuous in x and y. Furthermore f and g are locally Lipschitz in x and y.

(ii) $\|f(t,x,y)\| \le \alpha(y)\|x\| + c, \forall t \ge 0$

where c is a positive constant and $\alpha: R^m \to R^+$ is bounded for finite values of y.

(iii) There exists a functional $V: R^+ \times R^{n+m} \to R^+$ such that

$$k_1\|x\|^2 + k_2\|y\|^2 \le V(t,x,y) \le k_3\|x\|^2 + k_4\|y\|^2 \quad (3)$$

where k_1, k_2, k_3, k_4 are positive constants.

(iv) $$\dot{V}(t,x,y)\Big|_{(1)-(2)} \le -\beta(\|x\|) \quad (4)$$

where $\beta(\cdot)$ is a continuous monotone increasing function with $\beta(0) = 0$. Then $x(t) \to 0$ as $t \to \infty$.

Remark : Note that the state vector in the above theorem consists of two components (x, y). Condition (i) is basically for the local existence and uniqueness. Condition (iii) implies that $V(t)$ is positive definite and decrescent function. The right hand side of equation (4) involves only a part of state vector x, hence the equation (4) does not imply that $-\dot{V}(t)$ is (locally) positive definite function. Therefore the asymptotic stability can not be concluded. Lastly condition (ii) is additionally assumed in which y is treated as a parameter in f.

Proof of the Theorem: Defining $z = [x^T, y^T]^T$ with $z(0) = [x^T(0), y^T(0)]^T$, (1)-(2) can be rewritten as

$$\dot{z} = F(t,z), \quad z(0) = \begin{bmatrix} x_o \\ y_o \end{bmatrix}.$$

Since F is piecewise continuous in t and locally Lipschitz in z due to the condition (i), by the standard local existence theorem (see [1]) there exists a unique solution defined on an interval $J_T = [0, T)$ for some $T > 0$. Also the existence of a Lyapunov function V which is positive definite and decrescent and its derivative along (1)-(2) being ≤ 0 implies that a set $E_\gamma = \{z: V \le \gamma, \gamma \in R^+\}$ is positive invariant, i.e. $\|z(t)\| \le \gamma', \forall t \ge 0$, where γ' is a constant not depending on T.

Let the unique solution of (1) at time t starting with initial state $x(s)$ at initial time s be of the form

$$x(t) = x(s) + \int_s^t f(\tau, x(\tau), y(\tau))\, d\tau \quad (5)$$

and denote the solution as $x(t) = x(t, x(s), s)$. Then we can define a two parameter family of map $S(t,s)$ on R^n as

$$S(t,s)x(s) = x(t, x(s), s), \quad 0 \le s \le t < \infty. \quad (6)$$

Then by the uniqueness and continuous dependence of the solutions $x(t) = x(t, x(s), s)$ on the triple $(t, x(s), s)$, the mapping $S(t,s)$ on R^n becomes an evolution process such that [9, p.12]

(i) $S(\bullet, s)x(s): R^+ \to R^n$ is continuous (right continuous at $t = s$)

(ii) $S(t, \bullet)(\bullet): R \times R^n \to R^n$ is continuous

(iii) $S(s,s)x(s) = x(s)$

(iv) $S(t,s)x(s) = S(t,r)S(r,s)x(s)$, for all $x(s) \in R^n$ and $0 \le s \le r \le t < \infty$.

We further note that the condition (4) in the theorem implies that

$$\begin{aligned}\int_0^\infty \beta(\|S(t,0)x_o\|)\, dt &= \int_0^\infty \beta(\|x(t)\|)\, dt \\ &\le -\int_0^\infty \dot{V}(t,x,y)\, dt \quad (7) \\ &= V(0) - V(\infty) \\ &< \infty\end{aligned}$$

where $x(t) = S(t,0)x_0$.

Indeed, the conclusion of the theorem can be proven by contradiction. Suppose $S(t,0)x_o \not\to 0$ as $t \to \infty$, then there exists an $\varepsilon > 0$ and an infinite sequence $t_j \to \infty$ such that

$$\|S(t_j, 0)x_o\| \ge \varepsilon.$$

Now however small the ε is, there exist constants $M > 0$ and $\varepsilon_o > 0$ such that

$$M \ge \bar{\alpha} = \sup_{y \in E_\gamma} \alpha(y(t)),$$

and

$$\frac{\varepsilon}{e}-\frac{c}{M}\geq \varepsilon_o>0. \qquad (8)$$

Note that if $c=0$, then (8) is always satisfied. Therefore taking norms on both sides of (5)

$$\begin{aligned}\|x(t)\| &\leq \|x(s)\| + \int_s^t \left(\alpha(y(\tau))\|x(\tau)\|+c\right)d\tau \\ &\leq \|x(s)\| + \int_s^t \left(\sup_{y\in E_Y} \alpha(y(\tau))\|x(\tau)\|+c\right)d\tau \\ &\leq \|x(s)\| + \int_s^t M\left(\|x(\tau)\|+\frac{c}{M}\right)d\tau.\end{aligned}$$

Applying the Bellman-Gronwall's inequality yields

$$\|x(t)\| \leq \left(\|x(s)\|+\frac{c}{M}\right)e^{M(t-s)} \qquad (9)$$

for all $t\geq s\geq 0$.

Now without loss of generality we can assume that $t_{j+1}-t_j > 1/M$. If we set $\Delta_j=[t_j-1/M,\ t_j]$, then $m(\Delta_j)=1/M>0$ (m = Lebesgue measure) and the intervals Δ_j do not overlap. For $t\in\Delta_j$

$$\begin{aligned}\varepsilon &\leq \left\|S(t_j,0)x_o\right\| \\ &= \left\|S(t_j,t)S(t,0)x_o\right\| \\ &= \left\|S(t_j,t)x(t)\right\| \\ &\leq \left(\|x(t)\|+\frac{c}{M}\right)e^{M(t_j-t)} \\ &\leq \left(\|x(t)\|+\frac{c}{M}\right)e\end{aligned}$$

where the second inequality above is obtained from (9). Therefore we have

$$\begin{aligned}\|x(t)\| &\geq \frac{\varepsilon}{e}-\frac{c}{M} \\ &\geq \varepsilon_o\end{aligned}$$

for all $t\in\Delta_j=[t_j-1/M,\ t_j]$. Hence

$$\begin{aligned}\int_0^\infty \beta\left(\|S(t,0)x_o\|\right)dt &\geq \sum_{j=1}^{\infty}\int_{\Delta_j}\beta\left(\|S(t,0)x_o\|\right)dt \\ &\geq \sum_{j=1}^{\infty}\int_{\Delta_j}\beta(\varepsilon_o)\,dt \\ &= \beta(\varepsilon_o)\sum_{j=1}^{\infty}m(\Delta_j) \\ &= \infty\end{aligned}$$

contradicting (7). Thus we must have $x(t)\to 0$ as $t\to\infty$. Q.E.D.

Example: Consider a model reference adaptive control of a scalar differential equation. This example is well documented in adaptive control literature, for example [5, p.99] or [7, p.99]. Let the plant and reference model be

$$\begin{aligned}\dot{x}_p(t) &= a_p x_p(t) + k_p u(t) \\ \dot{x}_m(t) &= a_m x_m(t) + k_m r(t)\end{aligned}$$

where subscripts p and m denote plant and model, respectively, a_p and k_p are unknown parameters of the plant, $u(t)$is the control input, $a_m<0$, $r(t)$ is a bounded reference input. The control objective is to design a bounded control input $u(t)$ which enables the plant output $x_p(t)$ to follow the signal $x_m(t)$ generated from the reference model. The whole closed loop adaptive system results in the following equations (see the references cited above) as

$$\dot{x}(t) = \left(a_m + k_p y_1(t)\right)x(t) + k_p x_m(t) y_1(t) + k_p r(t) y_2(t) \qquad (9)$$

$$\dot{y}_1(t) = -sgn(k_p)x(t)\left(x(t)+x_m(t)\right) \qquad (10)$$

$$\dot{y}_2(t) = -sgn(k_p)x(t)r(t) \qquad (11)$$

where $x(t)\equiv x_p(t)-x_m(t)$, $y_1(t)$ and $y_2(t)$ are adjustable parameters in the controller, and $r(t)$ and $x_m(t)$ are treated to be bounded exogenous signals. Obtaining (10) and (11), a Lyapunov function $V: R^3\to R^+$ as

$$V(x,y_1,y_2)=\frac{1}{2}\left(x^2+|k_p|\left(y_1{}^2+y_2{}^2\right)\right) \qquad (12)$$

has been considered. The differentiation of V with respect to t along the equations (9)-(11) yields

$$\dot{V}(x,y_1,y_2)\Big| = a_m x^2(t)\leq 0. \qquad (13)$$

Note that equations (9), (10)-(11), (12) and (13) correspond to the equations (1), (2), (3) and (4), respectively, in the theorem and satisfy all the assumptions. Therefore by applying the theorem $\lim_{t\to\infty} x(t)=0$ can be shown.

Remark: The convergence of $x(t)$ to 0 as $t\to\infty$ in adaptive control literature (see [5, p.99] or [7, p.99]) has been shown by applying the Barbalat's Lemma [11, p.211].

3. CONCLUSIONS

In this note asymptotic convergence to zero of the partial state of coupled nonautonomous nonlinear system with immanent uniform stability has been shown. Considering that the invariance principle does not hold for general nonautonomous system and that it is sometimes difficult to obtain $-\dot{V}(t)$ to be positive definite, the obtained results could be useful in analysis of nonautonomous system since it can assert at least partial convergence of the state.

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