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NEW ANALYSIS OF NONLINEAR SYSTEM WITH TIME VARYING PARAMETER

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Abstracts In this paper, the frozen time approach is used to analyze the nonlinear system with time varying parameter. Using the extended linearization, we propose two analytical methods that compute an upper bound of the Euclidean norm of the difference between state variable and equilibrium point of the given system. The properties of the two methods are discussed with simple examples.

Keywords Frozen-time, Time varying parameter, Extended linearization, Norm bound, Kronecker sum

1. INTRODUCTION

Generally, most engineering systems are modelled by nonlinear equations. So because the dynamic properties change according as operating region changes, the classical methods can't satisfy the desired performance of given nonlinear system. Kelemen and Rugh presented stability results that deal with response of a nonlinear system to slowly varying input signals by use of the extended linearization method[1],[2]. But they could not present the norm bound of the difference between state variable and the parameterized equilibrium point. Recently, the norm bound was computed but it used a constant value which is not definitely defined, and its norm bound is larger than that of this paper.

This paper proposes two analytical methods for computing the norm bound. First method gives very small norm bound values, but it is not adequate for the higher ordered and complicate nonlinear systems. Another disadvantage is that the norm bound is related only to the system eigenvalues. Second method introducing the Kronecker sum does not need to compute matrix exponential function, so it can reduces the burdens of calculation. Moreover it relates the norm bound not only with system eigenvalues but also with system dimension. But unfortunately it has a little larger norm bound than the former. Finally, some simple examples are given to verify the results of this paper.

2. PRELIMINARIES

Given an $(n\times n)$ matrix $A=\{a_{ij}\}$, $\lambda_i(A)$ denotes the ith eigenvalue and $\sigma_i(A)=\sqrt{\lambda_i(A^TA)}$ the ith singular value of A; the maximum(minimum) singular value is denoted by $\sigma_{\max}(A)(\sigma_{\min}(A))$. $\|A\|_F=\sqrt{\text{trace}(A^TA)}$ indicates the Frobenius norm of A and $\|A\|_2=\sigma_{\max}(A)$ indicates the Euclidean matrix norm. Unless $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are mentioned, the symbol $\|A\|$ will be used to demote $\|A\|_2$. The n^2-vector $Vec[A]$ is the vector composed of the columns of matrix A taken in order. Given an $(n\times n)$ matrix $B=\{b_{ij}\}$, $A\otimes B=\{a_{ij}B\}$ is the Kronecker product and $A\oplus B=A\otimes I+I\otimes B$ the Kronecker sum, where I is the $n\times n$ identity matrix.

Consider some mathematical preliminaries.

($P1$) For any nonsingular matrix M, one has

$$\|M^{-1}\|=\frac{1}{\sigma_{\min}(M)}$$

($P2$) For any matrix S, one has

$$\|S\|_F=\|Vec[S]\|$$

($P3$) If $\{\lambda_i\}$ and $\{\mu_j\}$ are the eigenvalues of A and B respectively, then $\{\lambda_i+\mu_j\}$ are the eigenvalues of $A\oplus B$.

3. FORMULATION

The system is described by

$$\dot{x}(t)=f(x(t),u(t)),\quad x(0)=x_0,\quad t\geq 0 \tag{1}$$

where $x(t)$ is the $n\times 1$ state vector and $u(t)$ is the $m\times 1$ time varying parameter. We assume that

($H1$) $f:R^n\times R^m\rightarrow R^n$ is twice continuously differentiable.

($H2$) there is a bounded, open set $\Gamma\subset R^m$ and a continuously differentiable function $\mathrm{x}:\bar{\Gamma}\rightarrow R^n$ such that for each constant time varying parameter $\mathrm{w}\in\Gamma$, $f(\mathrm{x}(\mathrm{w}),\mathrm{w})=0$.

($H3$) there are $\sigma_0>0$, $k_1>0$, and $k_2>0$ such that for $\mathrm{w}\in\Gamma$,

$$Re\,\lambda_i\{\partial f(\mathrm{x}(\mathrm{w}),\mathrm{w})/\partial x\}\leq-\sigma_0$$

$$k_1\sigma_0\leq|\lambda_i\{\partial f(\mathrm{x}(\mathrm{w}),\mathrm{w})/\partial x\}|\leq k_2\sigma_0,\quad i=1,2,\ldots,n$$

($H4$) the continuously differentiable time varying parameter $u(t)$ is norm-bounded by $\delta>0$;

$$\|\dot{u}(t)\|\leq\delta$$

For notational convenience, we let $q(t)=\mathrm{x}(u(t))$, where $q(t)$ is the equilibrium trajectory, i.e.,the extended equilibrium point. (1) can be rewritten as[2]

$$\dot{e}(t)=A(t)e(t)+R(t)e(t)+B(t)\dot{u}(t) \tag{2}$$

where

$$\begin{aligned} e(t) &= x(t) - q(t) \\ A(t) &= \partial f(x,w)/\partial x\,|_{x=q} \\ B_w(t) &= \partial f(x,w)/\partial w\,|_{x=q} \\ B(t) &= A^{-1}(t)B_w(t) \\ R(t) &= \hat{R}(q,w,x) \\ &= \int_0^1 (\hat{A}(q+\theta(x-q),w) - \hat{A}(q,w))\,d\theta \end{aligned} \tag{3}$$

By $(H1)$ and $(H2)$, there exist positive constants K_A and K_B such that [2]

$$\|A(t)\| \le K_A, \quad \|B(t)\| \le K_B, \quad t \ge t_0 \tag{4}$$

4. METHOD- I

Lemma 1 If $\hat{A}(x,w)$ satisfies Lipschitz condition for $x \in D \subset R^n$, then there exists a finite constant L_A such that

$$\|\hat{R}(q,w,x)\| \le \frac{L_A}{2}\|e\| \tag{5}$$

Proof : Applying the Lipschitz condition gives

$$\begin{aligned} \|\hat{A}(q+\theta(x-q),w) - \hat{A}(q,w)\| &\le L_A\|(q+\theta(x-q))-q\| \\ &= L_A\|\theta(x-q)\| \\ &= \theta L_A\|e\| \end{aligned}$$

Therefore using (3),

$$\begin{aligned} \|\hat{R}(q,w,x)\| &\le \int_0^1 \|\hat{A}(q+\theta(x-q),w) - \hat{A}(q,w)\|\,d\theta \\ &\le \int_0^1 \theta L_A\|e\|\,d\theta \\ &= \frac{L_A}{2}\|e\| \end{aligned}$$

■

Consider $P(t)$ which transforms $A(t)$ into diagonal matrix $\tilde{A}(t)$. Using the frozen time approach, we can say that there exists $P(t)$ which satisfies $A(t) = P(t)\tilde{A}(t)P^{-1}(t)$ or $A(t)P(t) = P(t)\tilde{A}(t)$. Assume that there exists a constant $h_1 > 0$ such that

$$\begin{aligned} \|A(t)P(t)\| &\le \|A(t)\|\,\|P(t)\| \\ &\le h_1\|P(t)\|\,\|\tilde{A}(t)\| \end{aligned}$$

Therefore

$$\|A(t)\| \le h_1\|\tilde{A}(t)\| \tag{6}$$

can be obtained.

Lemma 2 There exists a finite constant K_A, such that

$$\|A(t)\| \le K_A \tag{7}$$

where

$$K_A = \begin{cases} h_1 k_2 \sigma_0 & : n_J = 1 \\ h_1(1+k_2\sigma_0) & : n_J \ge 2 \end{cases} \tag{8}$$

and n_J is the dimension of the largest Jordan block of $\tilde{A}(t)$.

Proof : i) For $n_J = 1$, $K_A = h_1 k_1 \sigma_0$ [3]

ii) For $n_J \ge 2$,

$$\tilde{A}(t) = \begin{pmatrix} \tilde{A}_1(t) & 0 & \cdots & 0 \\ 0 & \tilde{A}_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{A}_p(t) \end{pmatrix}, \quad 1 \le p \le n \tag{9}$$

where $\tilde{A}_i(t)$, $1 \le i \le p$ are Jordan blocks. Consider the maximum Jordan block $\tilde{A}_J(t)$,

$$\tilde{A}_J(t) = \begin{pmatrix} \tilde{a}_J(t) & 1 & 0 & \cdots & 0 \\ 0 & \tilde{a}_J(t) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \tilde{a}_J(t) \end{pmatrix} \tag{10}$$

Since the eigenvalue of $\tilde{A}_J(t)$ satisfies $k_1\sigma_0 \le |\tilde{a}_J(t)| \le k_2\sigma_0$ from $(H3)$, we can obtain

$$\begin{aligned} \|A(t)\| &\le h_1\|\tilde{A}(t)\| \\ &\le h_1\sqrt{\|\tilde{A}(t)\|_1\|\tilde{A}(t)\|_\infty} \\ &= h_1\sqrt{\|\tilde{A}_J(t)\|_1\|\tilde{A}_J(t)\|_\infty} \\ &\le h_1(1+k_2\sigma_0) \end{aligned}$$

■

Note that $A(t) = P(t)\tilde{A}(t)P^{-1}(t)$ can be rewritten as $A^{-1}(t)P(t) = P(t)\tilde{A}^{-1}(t)$, and assume that there exists a constant $h_2 > 0$, such that

$$\begin{aligned} \|A^{-1}(t)P(t)\| &\le \|A^{-1}(t)\|\,\|P(t)\| \\ &\le h_2\|P(t)\|\,\|\tilde{A}^{-1}(t)\| \end{aligned}$$

Therefore

$$\|A^{-1}(t)\| \le h_2\|\tilde{A}^{-1}(t)\| \tag{11}$$

can be obtained.

Lemma 3 There exists a finite constant K_B, such that

$$\|B(t)\| \le K_B \tag{12}$$

where

$$K_B = h_2 W_B \sum_{i=1}^{n_J}\left(\frac{1}{k_1\sigma_0}\right)^i \tag{13}$$

and

$$\|B_w(t)\| \le W_B \tag{14}$$

Proof : i) For $n_J = 1$, $K_B = \frac{h_2 W_B}{k_1\sigma_0}$ [3]

ii) For $n_J \ge 2$

The inverse matrix of the maximum Jordan block of $\tilde{A}(t)$ is

$$\tilde{A}_J^{-1}(t) = \begin{pmatrix} \frac{1}{\tilde{a}_J(t)} & \frac{-1}{(\tilde{a}_J(t))^2} & \cdots & \frac{(-1)^{n_J+1}}{(\tilde{a}_J(t))^{n_J}} \\ 0 & \frac{1}{\tilde{a}_J(t)} & \cdots & \frac{(-1)^{n_J}}{(\tilde{a}_J(t))^{n_J-1}} \\ 0 & 0 & \cdots & \frac{(-1)^{n_J-1}}{(\tilde{a}_J(t))^{n_J-2}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\tilde{a}_J(t)} \end{pmatrix} \tag{15}$$

Using $\frac{1}{k_2\sigma_0} \le \left|\frac{1}{\tilde{a}_J(t)}\right| \le \frac{1}{k_1\sigma_0}$ obtained from $(H3)$,

$$\begin{aligned} \|A^{-1}(t)\| &\le h_2\|\tilde{A}^{-1}(t)\| \\ &\le h_2\sqrt{\|\tilde{A}^{-1}(t)\|_1\|\tilde{A}^{-1}(t)\|_\infty} \\ &= h_2\sqrt{\|\tilde{A}_J^{-1}(t)\|_1\|\tilde{A}_J^{-1}(t)\|_\infty} \\ &\le h_2\sum_{i=1}^{n_J}\left(\frac{1}{k_1\sigma_0}\right)^i \end{aligned}$$

Finally using (3),

$$\begin{aligned} \|B(t)\| &\le \|A^{-1}(t)\|\,\|B_w(t)\| \\ &\le h_2 W_B \sum_{i=1}^{n_J}\left(\frac{1}{k_1\sigma_0}\right)^i \end{aligned}$$

■

The following quadratic form will be used extensively in the sequel. Let

$$V(t, e(t)) = e^T(t)Q(t)e(t) \tag{16}$$

$$Q(t) = \int_0^\infty e^{A^T(t)\tau}e^{A(t)\tau}d\tau \tag{17}$$

Note that $Q(t)$ is well defined, continuously differentiable, unique positive definite solution of

$$A^T(t)Q(t) + Q(t)A(t) = -I \tag{18}$$

Taking the exponential operation of $A(t) = P(t)\tilde{A}(t)P^{-1}(t)$ gives $e^{A(t)\tau}P(t) = P(t)e^{\tilde{A}(t)\tau}$. Assume that there exists a constant $h_3 > 0$, such that

$$\begin{aligned} \|e^{A(t)\tau}P(t)\| &\le \|e^{A(t)\tau}\|\,\|P(t)\| \\ &\le h_3\|P(t)\|\,\|e^{\tilde{A}(t)\tau}\| \end{aligned}$$

Therefore

$$\|e^{A(t)\tau}\| \le h_3\|e^{\tilde{A}(t)\tau}\| \tag{19}$$

can be obtained.

Lemma 4 There exist finite constant $M_1 \geq \mu_1 \geq 0$, such that

$$\mu_1 \| e(t) \|^2 \leq V(t, e(t)) \leq M_1 \| e(t) \|^2 \quad (20)$$

where

$$\mu_1 = \frac{1}{2K_A} \quad (21)$$
$$M_1 = h_3^2 \sum_{i=1}^{n_j} \sum_{j=1}^{n_j} \frac{(i+j-2)!}{(i-1)!\,(j-1)!\,(2\sigma_0)^{i+j-1}}$$

Proof : I) $\mu_1 = \frac{1}{2K_A}$ [2]

II) i) For $n_j = 1$, $M_1 = \frac{h_3^2}{2\sigma_0}$ [3]

ii) For $n_j \geq 2$,

The exponential of the maximum Jordan block of $\tilde{A}(t)$ is,

$$e^{\tilde{A}_j(t)\tau} = \begin{pmatrix} e^{\tilde{a}_j(t)\tau} & \tau e^{\tilde{a}_j(t)\tau} & \cdots & \frac{\tau^{(n_j-1)} e^{\tilde{a}_j(t)\tau}}{(n_j-1)!} \\ 0 & e^{\tilde{a}_j(t)\tau} & \cdots & \frac{\tau^{(n_j-2)} e^{\tilde{a}_j(t)\tau}}{(n_j-2)!} \\ 0 & 0 & \cdots & \frac{\tau^{(n_j-3)} e^{\tilde{a}_j(t)\tau}}{(n_j-3)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\tilde{a}_j(t)\tau} \end{pmatrix} \quad (22)$$

Since $Re\{\ \tilde{a}_j(t)\ \} \leq -\sigma_0$ from $(H3)$, therefore

$$\begin{aligned} \| e^{A(t)\tau} \| &\leq h_3 \| e^{\tilde{A}(t)\tau} \| \\ &\leq h_3 \sqrt{\| e^{\tilde{A}(t)\tau} \|_1 \| e^{\tilde{A}(t)\tau} \|_\infty} \\ &= h_3 \sqrt{\| e^{\tilde{A}_j(t)\tau} \|_1 \| e^{\tilde{A}_j(t)\tau} \|_\infty} \\ &\leq h_3 e^{-\sigma_0 \tau} \sum_{i=1}^{n_j} \frac{\tau^{i-1}}{(i-1)!} \end{aligned}$$

Finally,

$$\begin{aligned} V(t, e(t)) &\leq \| Q(t) \| \; \| e(t) \|^2 \\ &\leq \int_0^\infty \| e^{A(t)\tau} \|^2 d\tau \| e(t) \|^2 \\ &\leq h_3^2 \int_0^\infty e^{-2\sigma_0 \tau} \sum_{i=1}^{n_j} \sum_{j=1}^{n_j} \frac{\tau^{i+j-2}}{(i-1)!(j-1)!} d\tau \| e(t) \|^2 \\ &= h_3^2 \sum_{i=1}^{n_j} \sum_{j=1}^{n_j} \frac{(i+j-2)!}{(i-1)!(j-1)!(2\sigma_0)^{i+j-1}} \| e(t) \|^2 \end{aligned}$$

■

Theorem 1 Suppose the system satisfies $(H1), \ldots, (H4)$. Then $\| e(t) \|$ satisfies

$$\| e(t) \| \leq \| e(0) \| \sqrt{\frac{M_1}{\mu_1}} e^{-\frac{N_1}{2M_1}t} + a_1, \quad t \geq 0 \quad (23)$$

if following conditions are satisfied

$$\begin{aligned} &(1)\ \| e(0) \| < a_2 \\ &(2)\ 1 - 2\delta M_1^2 K_D > 0 \\ &(3)\ -8\delta K_B L_A M_1^2 + \theta^2 (1 - 4\delta K_D M_1^2 + 4\delta^2 K_D^2 M_1^4) > 0 \end{aligned} \quad (24)$$

Proof : Differentiating $V(t, e(t))$ with respect to t and appling (1) give

$$\begin{aligned} \dot{V}(t, e(t)) = & -e^T(t) e(t) + e^T(t) \dot{Q}(t) e(t) \\ & + 2e^T(t) Q(t) R(t) e(t) + 2e^T(t) Q(t) B(t) \dot{w}(t) \end{aligned}$$

Using Lemma 1, $\| \dot{A}(t) \| \leq K_D \| \dot{w}(t) \|$ [2], $\| \dot{Q}(t) \| \leq 2 M_1^2 K_D \| \dot{w}(t) \|$ [2], and $\| Q(t) \| \leq M_1$, we obtain

$$\begin{aligned} \dot{V}(t, e(t)) \leq & - \| e(t) \|^2 + \| \dot{Q}(t) \| \; \| e(t) \|^2 \\ & + 2 \| Q(t) \| \; \| R(t) \| \; \| e(t) \|^2 \\ & + 2 \| Q(t) \| \; \| B(t) \| \; \| e(t) \| \; \| \dot{w}(t) \| \\ \leq & -(1 - 2\delta M_1^2 K_D) \| e(t) \|^2 + M_1 L_A \| e(t) \|^3 \\ & + 2\delta M_1 K_B \| e(t) \| \\ = & -(1-\theta)(1 - 2\delta M_1^2 K_D) \| e(t) \|^2 + M_1 L_A \| e(t) \|^3 \\ & - \theta (1 - 2\delta M_1^2 K_D) \| e(t) \|^2 + 2\delta M_1 K_B \| e(t) \| \\ = & -N_1 \| e(t) \|^2 + W_1(e(t)) \end{aligned}$$

where $0 < \theta < 1$ and

$$\begin{aligned} N_1 &= (1-\theta)(1 - 2\delta M_1^2 K_D) \\ W_1(e(t)) &= M_1 L_A \| e(t) \|^3 - \theta (1 - 2\delta M_1^2 K_D) \| e(t) \|^2 \\ &\quad + 2\delta M_1 K_B \| e(t) \| \end{aligned} \quad (25)$$

If $N_1 > 0$ and $W_1(e(t)) \leq 0$ are guaranteed, then it always satisfies

$$\dot{V}(t, e(t)) \leq -N_1 \| e(t) \|^2 \quad (26)$$

Therefore, $W_1(e(t))$ can be rewritten as

$$W_1(e(t)) = \| e(t) \| (\| e(t) \| - a_1)(\| e(t) \| - a_2) \quad (27)$$

where

$$a_{1,2} = \frac{\theta(1 - 2\delta M^2 K_D) \mp \sqrt{-8\delta K_B L_A M^2 + \theta^2 (1 - 4\delta K_D M^2 + 4\delta^2 K_D^2 M^4)}}{2 L_A M}$$

If $(1 - 2\delta M_1^2 K_D) > 0$ and $\{ -8\delta K_B L_A M_1^2 + \theta^2 (1 - 4\delta K_D M_1^2 + 4\delta^2 K_D^2 M_1^4) \} > 0$ are satisfied, then a_1 and a_2 will be guaranteed to be positive real constants and $\lim_{t \to \infty} \| e(t) \|$ will be equal to or less than a_1 for $\| e(0) \| < a_2$. From (20) and (26)

$$\mu_1 \| e(t) \|^2 \leq V(t, e(t)) \leq V(0, e(0)) e^{-\frac{N_1}{M_1}t} \quad (28)$$

can be obtained[4]. Finally, by comparing (20) with (28) we can obtain

$$\mu_1 \| e(t) \|^2 \leq V(t, e(t)) \leq M_1 \| e(0) \|^2 e^{-\frac{N_1}{M_1}t} \quad (29)$$

$$\| e(t) \| \leq \| e(0) \| \sqrt{\frac{M_1}{\mu_1}} e^{-\frac{N_1}{2M_1}t} \quad (30)$$

We can guarantee (30) only if $\lim_{t \to \infty} \| e(t) \| \leq a_1$ is satisfied. So generally we can write the norm bound of $e(t)$ as

$$\| e(t) \| \leq \| e(0) \| \sqrt{\frac{M_1}{\mu_1}} e^{-\frac{N_1}{2M_1}t} + a_1, \quad t \geq 0 \quad (31)$$

■

5. METHOD-II : Kronecker Sum

Lemma 5 There exists a finite constant K_C, such that

$$\| (A(t) \oplus A(t))^{-1} \| \leq K_C \quad (32)$$

where

$$K_C = h_4 \sum_{i=1}^{n_j} \left(\frac{1}{2k_1 \sigma_0} \right)^i \quad (33)$$

Proof : Define

$$H(t) = A(t) \oplus A(t) \quad (34)$$

By a frozen time approach, the matrix $H(t)$ can be transformed into $\tilde{H}(t) = P^{-1}(t) H(t) P(t)$, and it follows $H^{-1}(t) P(t) = P(t) \tilde{H}^{-1}(t)$. Assume that there exists a finite constant $h_4 > 0$ which satisfies

$$\begin{aligned} \| H^{-1}(t) P(t) \| &= \| H^{-1}(t) \| \; \| P(t) \| \\ &\leq h_4 \| P(t) \| \; \| \tilde{H}^{-1}(t) \| \end{aligned}$$

Therefore it follows

$$\| H^{-1}(t) \| \leq h_4 \| \tilde{H}^{-1}(t) \| \quad (35)$$

i) For $n_j = 1$,

$$\tilde{H}^{-1}(t) = \begin{pmatrix} \frac{1}{\tilde{h}_1(t)} & 0 & \cdots & 0 \\ 0 & \frac{1}{\tilde{h}_2(t)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\tilde{h}_n(t)} \end{pmatrix} \quad (36)$$

where $\tilde{h}_i(t)$, $i = 1, \ldots, n$ are eigenvalues of $\tilde{H}(t)$. Using $2k_1\sigma_0 \leq |\ \tilde{h}_i(t)\ | \leq 2k_2\sigma_0$ obtained from $(H3)$ and $(P3)$,

$$\begin{aligned} \| H^{-1}(t) \| &= h_4 \| \tilde{H}^{-1}(t) \| \\ &= h_4 \sqrt{\lambda_{\max} \{ (\tilde{H}^{-1}(t))^T \tilde{H}^{-1}(t) \}} \\ &\leq \frac{h_4}{2k_1 \sigma_0} \end{aligned}$$

ii) For $n_j \geq 2$,

The inverse matrix of the maximum Jordan block of $\tilde{H}(t)$ is

$$\hat{H}_j^{-1}(t)=\begin{pmatrix} \frac{1}{\hat{h}_j(t)} & \frac{-1}{(\hat{h}_j(t))^2} & \cdots & \frac{(-1)^{n_j+1}}{(\hat{h}_j(t))^{n_j}} \\ 0 & \frac{1}{\hat{h}_j(t)} & \cdots & \frac{(-1)^{n_j}}{(\hat{h}_j(t))^{n_j-1}} \\ 0 & 0 & \cdots & \frac{(-1)^{n_j-1}}{(\hat{h}_j(t))^{n_j-2}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\hat{h}_j(t)} \end{pmatrix} \quad (37)$$

Finally,

$$\begin{aligned} \| H^{-1}(t) \| &\le h_4 \| \hat{H}^{-1}(t) \| \\ &\le h_4 \sqrt{\| \hat{H}^{-1}(t) \|_1 \| \hat{H}^{-1}(t) \|_\infty} \\ &= h_4 \sqrt{\| \hat{H}_j^{-1}(t) \|_1 \| \hat{H}_j^{-1}(t) \|_\infty} \\ &\le h_4 \sum_{i=1}^{n_j} (\frac{1}{2k_1\sigma_0})^i \end{aligned}$$

■

Lemma 6 There exist finite constants $M_2 \ge \mu_2 \ge 0$, such that

$$\mu_2 \| e(t) \|^2 \le V(t, e(t)) \le M_2 \| e(t) \|^2$$

where

$$\begin{aligned} \mu_2 &= \frac{1}{2K_A} \\ M_2 &= \sqrt{n} K_C \end{aligned} \quad (38)$$

Proof :

Ⅰ) Using (16), (18), $e^T(t)Q(t)e(t) \ge \lambda_{\min}(Q(t)) \| e(t) \|^2$, and $\sigma_{\min}(Q(t)) \ge \frac{1}{2\| A(t) \|}$ [5],

$$\begin{aligned} V(t, e(t)) &\ge \lambda_{\min}(Q(t)) \| e(t) \|^2 \\ &\ge \frac{1}{2K_A} \| e(t) \|^2 \end{aligned}$$

Define

$$\mu_2 = \frac{1}{2K_A}$$

Ⅱ) (18) can be rewritten as[5],

$$Vec[\,Q(t)\,] = -(A^T(t) \oplus A^T(t))^{-1} Vec[\,I\,] \quad (39)$$

Using mathematical preliminaries and Lemma 5,

$$\begin{aligned} \| Q(t) \| &\le \| Q(t) \|_F \\ &= \| Vec[\,Q(t)\,] \| \\ &\le \| (A^T(t) \oplus A^T(t))^{-1} \| \, \| I \|_F \\ &= \sqrt{n} K_C \end{aligned}$$

and

$$\begin{aligned} V(t, e(t)) &\le \| Q(t) \| \, \| e(t) \|^2 \\ &\le \sqrt{n} K_C \| e(t) \|^2 \end{aligned}$$

Therefore we can obtain

$$M_2 = \sqrt{n} K_C$$

■

Theorem 2 Suppose the system satisfies $(H1), \ldots, (H4)$. Then $\| e(t) \|$ satisfies

$$\| e(t) \| \le \| e(0) \| \sqrt{\frac{M_2}{\mu_2}} e^{-\frac{N_2}{2M_2}t} + b_1, \quad t \ge 0 \quad (40)$$

if following conditions are satisfied

(1) $\| e(0) \| < b_2$

(2) $1 - 2\sqrt{n} K_C^2 K_{DA} > 0$

(3) $-8n\delta L_A K_B K_C^2 + \theta^2(1 - 4\sqrt{n} K_C^2 K_{DA} + 4n K_C^4 K_{DA}^2) > 0$

Proof : Assume that there exists a finite constant K_{DA}, such that

$$\| \dot{A}(t) \| \le K_{DA} \quad (41)$$

By using Lemma 5 and (41), $\| \dot{Q}(t) \| \le 2\sqrt{n} \| (A(t) \oplus A(t))^{-1} \|^2 \| \dot{A}(t) \|$ [5] can be rewritten as

$$\| \dot{Q}(t) \| \le 2\sqrt{n} K_C^2 K_{DA} \quad (42)$$

Since $\| Q(t) \| \le \sqrt{n} K_C$ and $\| \dot{Q}(t) \| \le 2\sqrt{n} K_C^2 K_{DA}$,

$$\begin{aligned} \dot{V}(t, e(t)) &= -e^T(t)e(t) + e^T(t)\dot{Q}(t)e(t) + 2e^T(t)Q(t)R(t)e(t) \\ &\quad + 2e^T(t)Q(t)B(t)w(t) \\ &\le -\| e(t) \|^2 + \| \dot{Q}(t) \| \, \| e(t) \|^2 \\ &\quad + 2\| Q(t) \| \, \| R(t) \| \, \| e(t) \|^2 \\ &\quad + 2\| Q(t) \| \, \| B(t) \| \, \| e(t) \| \, \| w(t) \| \\ &\le -(1 - 2\sqrt{n} K_C^2 K_{DA}) \| e(t) \|^2 + \sqrt{n} K_C L_A \| e(t) \|^3 \\ &\quad + 2\sqrt{n}\delta K_C K_B \| e(t) \| \\ &= -(1-\theta)(1 - 2\sqrt{n} K_C^2 K_{DA}) \| e(t) \|^2 \\ &\quad \sqrt{n} K_C L_A \| e(t) \|^3 - \theta(1 - 2\sqrt{n} K_C^2 K_{DA}) \| e(t) \|^2 \\ &\quad + 2\sqrt{n}\delta K_C K_B \| e(t) \| \\ &= -N_2 \| e(t) \|^2 + W_2(e(t)) \end{aligned}$$

As the same way,

$$N_2 = (1-\theta)(1 - 2\sqrt{n} K_C^2 K_{DA}) \quad (43)$$

and

$$\begin{aligned} W_2(e(t)) &= \sqrt{n} K_C L_A \| e(t) \|^3 - \theta(1 - 2\sqrt{n} K_C^2 K_{DA}) \| e(t) \|^2 \\ &\quad + 2\sqrt{n}\delta K_C K_B \| e(t) \| \\ &= \| e(t) \| (\| e(t) \| - b_1)(\| e(t) \| - b_2) \end{aligned}$$

where

$$\begin{aligned} b_{1,2} &= \frac{\theta(1 - 2\sqrt{n} K_C^2 K_{DA})}{2\sqrt{n L_A K_C}} \\ &\mp \frac{\sqrt{-8n\delta L_A K_B K_C^2 + \theta^2(1 - 4\sqrt{n} K_C^2 K_{DA} + 4n K_C^4 K_{DA}^2)}}{2\sqrt{n L_A K_C}} \end{aligned}$$

The main idea of this proof has been illustrated so far, and the remaining part of it is the same as that of Theorem 1. ■

6. EXAMPLE

We introduce a simple example that will illustrate results of this paper. Consider the plant given by

$$\begin{aligned} \dot{x}_1(t) &= -10x_1(t) + x_2^2(t) + w(t) \\ \dot{x}_2(t) &= x_1^2(t) - mx_2(t) - \frac{w^2(t)}{100} \end{aligned} \quad (44)$$

where $w(t) = \sin(2t/3)$ and m is a constant value in the range of $5 \le m \le 10$.. The plant satisfies $(H1)$-$(H4)$, and the plant's family of equilibrium point is given by $x_1(w) = w/10$ and $x_2(w) = 0$. Table.1 and Table.2 show the results of analysis by method-Ⅰ and method-Ⅱ. We can confirm that method 1 gives the smaller norm bound of the steady state error.

TABLE 1. Result by method-Ⅰ ($\theta = 0.99$)

m	K_B	K_D	M	a_1	a_2
5	0.2001	0.2000	0.1000	0.0272	4.9096
6	0.1667	0.2000	0.0834	0.0188	5.9054
7	0.1429	0.2000	0.0715	0.0138	6.8998
8	0.1251	0.2000	0.0625	0.0106	7.9012
9	0.1112	0.2000	0.0556	0.0083	8.8872
10	0.0556	0.2000	0.0221	0.0017	22.3936

TABLE 2. Result by method-Ⅱ ($\theta = 0.99$)

m	K_{DA}	K_D	$1/K_C$	b_1	b_2
5	0.1333	0.2001	9.9968	0.0387	3.4472
6	0.1333	0.1667	11.9957	0.0267	4.1610
7	0.1333	0.1429	13.9940	0.0196	4.8692
8	0.1333	0.1251	15.9906	0.0150	5.5738
9	0.1333	0.1112	17.9807	0.0118	6.2744
10	0.1333	0.0556	19.8005	0.0054	6.9185

7. CONCLUSION

The analysis results discussed here present the frozen time approach to compute the norm bound of the difference between state variable and the parameterized equilibrium point. That is to say, the norm bound can be composed of both a exponentially decaying term and a constant term. Method-I shows the norm bound is related only with system eigenvalues, but that of method-II has more information, i.e., not only system eigenvalues but also system dimension. Another difference is that the former has smaller norm bound than the latter, but the latter need not to calculate $\| e^{A(t)\tau} \|$ which will be difficult to compute if the system complexity increases. More study is need in reducing the norm bound precisely and relating it with more system information.

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