Receding Horizon $H_{\infty}$ Predictive Control for Linear State-delay Systems

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Abstract: This paper proposes the receding horizon $H_{\infty}$ predictive control (RHHPC) for systems with a state-delay. We first propose a new cost function for a finite horizon dynamic game problem. The proposed cost function includes two terminal weighting terms, each of which is parameterized by a positive definite matrix, called a terminal weighting matrix. Secondly, we derive the RHHPC from the solution to the finite dynamic game problem. Thirdly, we propose an LMI condition under which the saddle point value satisfies the well-known nonincreasing monotonicity. Finally, we shows the asymptotic stability and $H_{\infty}$-norm boundedness of the closed-loop system controlled by the proposed RHHPC. Through a numerical example, we show that the proposed RHHPC is stabilizing and satisfies the infinite horizon $H_{\infty}$-norm bound.

Keywords: Receding horizon $H_{\infty}$ control, Terminal weighting matrix, Nonincreasing monotonicity, $H_{\infty}$-norm bound.

1. Introduction

Receding horizon controls (RHC) have attracted much attention from academia and industry because of its ability to handle time-varying systems, disturbances, input constraint, time-varying tracking commands, and nonlinear systems. Most research results about the RHC are concentrated on ordinary systems without delays. However, RHC approaches for systems with a state-delay can be found only recently [1], [2]. In [1], a simple control method based on the receding horizon concept is proposed. However, it does not have a state weighting in the cost function. Furthermore, it does not guarantee closed-loop stability at the stage when RHC solution is obtained. The stability can be checked only after the RHC solution is obtained. General extension of the RHC to state-delayed systems is proposed in [2].

The RHC for ordinary systems has been applied to $H_{\infty}$ problems recently in order to combine the practical advantage of the RHC with the robustness of the $H_{\infty}$ control [3]-[6]. The saddle point value in $H_{\infty}$ problems corresponds to the optimal cost in LQ problems. Those results in [3]-[6] propose conditions for nonincreasing monotonicity of the saddle point value. For state-delayed systems, there have been many approaches for $H_{\infty}$ problems [7], [8], [9]. Among them, the results in [8] consider a finite horizon $H_{\infty}$ control problems. However, since it deals with finite horizon problem, the closed-loop stability issue was not dealt with.

In this paper, we make the first approach to extend the receding horizon $H_{\infty}$ predictive control (RHHPC) to state-delayed systems. The proposed RHHPC has asymptotic closed-loop stability and satisfies the infinite horizon $H_{\infty}$-norm bound. We will propose a condition for nonincreasing monotonicity of the optimal cost, which plays an important role for stability and $H_{\infty}$-norm boundedness.

2. Receding Horizon $H_{\infty}$ Control

Consider a linear time-invariant system with a state-delay

$$
\begin{align*}
x(t) &= Ax(t) + A_1x(t-h) + Bu(t) + B_uw(t) \\
z(t) &=Cx(t) + Du(t)
\end{align*}
$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the control, $w \in \mathbb{R}^l$ the disturbance, $z \in \mathbb{R}^q$ the controlled output, $h > 0$ the delay, respectively. It is assumed that $C^TD = 0$ and $D^TD = I$. In order to obtain an RHHPC, we will first consider the following cost function with the finite horizon

$$
J(x_{t_0}, t_0, t_f, u, w) = \int_{t_0}^{t_f} \left[ x^T(t)Qx(t) + u^T(t)u(t) - \gamma^2 w^T(t)w(t) \right] dt + x^T(t_f)F_1x(t_f) + \int_{t_f}^{t_f \ominus h} x^T(t)F_2x(t)dt,
$$

where $Q = C^TC > 0$, $F_1 > 0$, $F_2 > 0$. We can regard $J$ as a function of either $L_2$ signals of feedback strategies. Let $\mathcal{M} = \{ \mu : [t_0, t_f] \times C_a[-h, 0] \rightarrow \mathbb{R}^m \}$ and $\mathcal{N} = \{ \nu : [t_0, t_f] \times C_a[-h, 0] \rightarrow \mathbb{R}^l \}$, where $C_a[-h, 0]$ is the space of $n$-dimensional vector functions continuous on $[-h, 0]$. Spaces $\mathcal{M}$ and $\mathcal{N}$ are the strategy spaces, and we will write strategies as $\mu, \nu$ to distinguish them from signals $u, w$.

Denote $x_2 = x(t + \theta), \ h \in [-h, 0], \ x_1 \in C_a[-h, 0]$ by the definition of $C_a[-h, 0]$. Let’s formulate a dynamic game problem

$$
\min_{\mu \in \mathcal{M}} \max_{\nu \in \mathcal{N}} J(x_{t_0}, t_0, t_f, \mu, \nu),
$$

which is a zero sum game, where $u$ is the minimizing player and $w$ is the maximizing player. The optimal $u$ and the worst case $w$ are called saddle point strategies. A saddle point solution $u(\tau) = \mu^*(\tau, x(\tau)), w(\tau) = \nu^*(\tau, x(\tau))$ satisfies

$$
J(x_{t_0}, t_0, t_f, \mu^*, \nu^*, w) \leq J(x_{t_0}, t_0, t_f, \mu^*, \nu^*, \nu) \leq J(x_{t_0}, t_0, t_f, u, \nu^*)
$$

for all $u, w \in L_2[t_0, t_f]$. For simplicity, $J(x_{t_0}, t_0, t_f, \mu^*, \nu^*)$ will be denoted by $J(x_{t_0}, t_0, t_f)$. The purpose of this paper is to develop a method to design a control law $u, w$, based on the receding horizon concept such that

(a) in case of zero disturbance, the closed-loop system is asymptotically stable
(b) with zero initial condition, the closed-loop transfer function $T_{zw}$ satisfies the $H_{\infty}$-norm bound, for given $\gamma > 0$.

$$
\|T_{zw}\|_{\infty} \leq \gamma.
$$
Since the control is based on the receding horizon strategy and the closed-loop system satisfies the $H_\infty$-norm bound, we will call it receding horizon $H_\infty$ predictive control (RHHPC) throughout this paper.

**Remark 1:** It is noted that the terminal weighting function consists of two parameters, characterized by two matrices $F_1$ and $F_2$. We will call them terminal weighting matrices in this paper. The purpose of adding a second terminal weighting term, parameterized by $F_2$, is to take the delay effect into account such that the resulting RHHPC is stabilizing. More specifically, if $F_2$ is chosen properly, the saddle point value satisfy the well-known 'nonincreasing monotonicity property', which will be considered in Section 3.

Before moving on, we introduce a lemma, which gives the sufficient condition for the existence of a saddle point solution. In the lemma, $V(\tau, x_\tau) : [t_0, t_j] \times C_{n-\infty, 0} \rightarrow \mathbb{R}$ denotes a continuous and differentiable functional. Furthermore, we will use the notation

$$
\frac{d}{d\tau} V(\tau, x_\tau)|_{\tau = \tau'} = \lim_{\Delta \tau \to 0} \frac{V(\tau + \Delta \tau, x_{\tau + \Delta \tau}) - V(\tau, x_\tau)}{\Delta \tau}
$$

where $x_{\tau + \Delta \tau} = x(\tau + \Delta \tau, x_\tau), s \in [-h, 0]$ is the solution of the system (1) resulting from the control $u(t) = \mu(t, x_\tau)$ and disturbance $w(t) = w(t, x_\tau)$.

**Lemma 1:** If it is possible to find a continuous functional $V(\tau, x_\tau) : [t_0, t_j] \times C_{n-\infty, 0} \rightarrow \mathbb{R}$, and a vector functional $\mu(\tau, x_\tau) : [t_0, t_j] \times C_{n-\infty, 0} \rightarrow \mathbb{R}^m$ and $v(\tau, x_\tau) : [t_0, t_j] \times C_{n-\infty, 0} \rightarrow \mathbb{R}^n$ such that

\begin{equation}
\frac{d}{d\tau} V(\tau, x_\tau)|_{\tau = \tau'} + x^T(\tau) F_1 x(\tau) + \int_{t_j - h}^{t_j} x^T(\tau) F_2 x(\tau) d\tau
\end{equation}

\begin{equation}
\frac{d}{d\tau} V(\tau, x_\tau)|_{\tau = \tau'} + x^T(\tau) Q x(\tau) + \mu^T(\tau, x_\tau) mu(\tau, x_\tau) - \gamma^2 \nu^T(\tau, x_\tau) \nu(\tau, x_\tau) = 0
\end{equation}

From (a),

$$
V(s, x_s) = \int_{t_0}^{t_j} x^T(\tau) Q x(\tau) + \mu^T(\tau, x_\tau) \mu(\tau, x_\tau) - \gamma^2 \nu^T(\tau, x_\tau) \nu(\tau, x_\tau) d\tau = 0
$$

From (c),

$$
\frac{d}{d\tau} V(\tau, x_\tau)|_{\tau = \tau'} + \int_{t_j - h}^{t_j} x^T(\tau) F_2 x(\tau) d\tau = J(x_s, s, t_f, \mu^*, \nu^*)
$$

$$
\frac{d}{d\tau} V(\tau, x_\tau)|_{\tau = \tau'} + x^T(\tau) Q x(\tau) + \mu^T(\tau, x_\tau) \mu(\tau, x_\tau) - \gamma^2 \nu^T(\tau, x_\tau) \nu(\tau, x_\tau) \nu(\tau, x_\tau) = 0
$$

Integrating the above from $s$ to $t_f$ yields

\begin{equation}
V(s, x_s) \geq V(t_0, x_{t_0}) = J(x_{t_0}, t_0, t_f, \mu^*, \nu^*)
\end{equation}

Similarly, we have

If we take $s = t_0$, then

This completes the proof.

From the above lemma, we see that $V(\tau, x_\tau)$ is a saddle point value, that is, $V(\tau, x_\tau) = J^*(x_s, t_f, \mu^*, \nu^*)$. Furthermore, it is noted that $V(s, x_s) \geq 0$ for all $s \in [t_0, t_j]$. This can be verified from Lemma 1.

Before deriving the receding horizon predictive control, we first provide a solution to a finite horizon dynamic game problem. The derivation is based on the above lemma. The procedure taken for derivation of the solution is similar to that used in [2]. Readers are referred to [2]. We assume the saddle point value has the form

\begin{equation}
V(\tau, x_\tau) = \begin{cases}
\begin{array}{l}
x^T(\tau) P_1(\tau) x(\tau) + x^T(\tau) \int_{t_0}^{t} P_2(\tau, s) x(\tau + s) ds + \\
\int_{t_j - h}^{t} x^T(\tau) W_3(\tau, s) x(\tau + s) ds
\end{array}
\end{cases}
\end{equation}

for $0 \leq t < t_j - h$

\begin{equation}
\begin{cases}
\begin{array}{l}
x^T(\tau) W_1(\tau) x(\tau) + 2x^T(\tau) \int_{t_j - h}^{t} W_2(\tau, s) x(\tau + s) ds + \\
\int_{t_j - h}^{t} x^T(\tau) F_2 x(\tau) d\tau
\end{array}
\end{cases}
\end{equation}

for $t_j - h \leq \tau \leq t_j$. 

\begin{equation}
J(x_s, s, t_f, \mu^*, \nu^*) \leq J(x_s, s, t_f, \mu^*, \nu^*) \leq J(x_s, s, t_f, \mu^*, \nu^*)
\end{equation}
Using the above saddle point value, the saddle point strategies for the dynamic game problem in (3) are given by

$$
\nu^*(\tau, x_{-\tau}) = \begin{cases} 
-B^T P_1(\tau) x(\tau) + \int_{-\tau}^{0} P_2(\tau, s) x(\tau + s) ds \\
\text{for } t_0 \leq \tau < t_f - h.
\end{cases}
$$

and

$$
\nu^*(\tau, x_{-\tau}) = \begin{cases} 
\gamma^{-2} B_0^T \left[ P_1(\tau) x(\tau) + \int_{-\tau}^{0} P_2(\tau, s) x(\tau + s) ds \right] \\
\text{for } t_0 \leq \tau < t_f - h.
\end{cases}
$$

for $t_f - h \leq \tau \leq t_f$.

$P_1, P_2, \text{ and } P_3$ satisfy the following Riccati-type partial differential equations:

$$
\dot{P}_1(\tau) + A^T P_1(\tau) + P_1(\tau) A - P_1(\tau) (BB^T - \gamma^{-2} B_u B_u^T) P_1(\tau) + Q + P_2(\tau, 0) + P_2(\tau, 0) = 0
$$

(7)

$$
\left( \frac{\partial}{\partial \tau} - \frac{\partial}{\partial s} \right) P_2(\tau, s) + A^T P_2(\tau, s) + P_3(\tau, 0, s) - P_1(\tau)(BB^T - \gamma^{-2} B_u B_u^T) P_3(\tau, s) = 0
$$

(8)

$$
\left( \frac{\partial}{\partial \tau} - \frac{\partial}{\partial r} + \frac{\partial}{\partial s} \right) P_3(\tau, r, s) - P_2^T(\tau, r)(BB^T - \gamma^{-2} B_u B_u^T) P_3(\tau, s) = 0
$$

(9)

with boundary conditions

$$
P_2(\tau, -h) = P_1(\tau) A_1
$$

(10)

$$
P_3(\tau, -h, s) = A_1^T P_2(\tau, s),
$$

(11)

where $t_0 \leq \tau < t_f - h$, $-h \leq r \leq 0$ and $-h \leq s \leq 0$. Furthermore, $W_1, W_2, \text{ and } W_3$ satisfy the following Riccati-type partial differential equations:

$$
\dot{W}_1(\tau) + A^T W_1(\tau) + W_1(\tau) A - W_1(\tau)(BB^T - \gamma^{-2} B_u B_u^T) W_1(\tau) + Q + F_2 = 0
$$

(12)

$$
\left( \frac{\partial}{\partial \tau} - \frac{\partial}{\partial s} \right) W_2(\tau, s) + A^T W_2(\tau, s) - W_1(\tau)(BB^T - \gamma^{-2} B_u B_u^T) W_2(\tau, s) = 0
$$

(13)

$$
\left( \frac{\partial}{\partial \tau} - \frac{\partial}{\partial r} + \frac{\partial}{\partial s} \right) W_3(\tau, r, s) - W_2^T(\tau, r)(BB^T - \gamma^{-2} B_u B_u^T) W_3(\tau, s) = 0
$$

(14)

with boundary conditions

$$
W_2(\tau, -h) = W_1(\tau) A_1
$$

(15)

$$
W_3(\tau, -h, s) = A_1^T W_2(\tau, s),
$$

where $t_f - h \leq \tau < t_f$, $-h \leq r \leq 0$ and $-h \leq s \leq 0$. In addition, $P_i$ and $W_i$ satisfy the following boundary conditions:

$$
W_1(t_f) = F_1
$$

(12)

$$
P_1(t_f - h) = W_1(t_f - h)
$$

(13)

$$
P_3(t_f - h, s) = W_3(t_f - h, s)
$$

(14)

$$
P_3(t_f - h, r, s) = W_3(t_f - h, r, s).
$$

(15)

$P_1, P_2, P_3$ and $W_1, W_2, W_3$ are solved backward in time from $t_f$ to $t_0$. Because the system is time-invariant, the shape of $P_i$ and $W_i$ are only characterized by the difference between the initial time and the final time, that is, $t_f - t_0$. If $t_f - t_0$ varies, the values of $P_i$ and $W_i$ at the initial time, $t_0$, also vary. However, if $t_f - t_0$ is fixed to a constant value, the values are all the same at initial time. For example, $P_1(t_0)$ with $t_0 = 1$ and $t_f = 5$ is equal to $P_1(t_0)$ with $t_0 = 2$ and $t_f = 6$. If we take receding horizon strategy, $t_0$ and $t_f$ corresponds to $t$, current time, and $t + T_o$, respectively. Therefore, the difference between the initial time and the terminal time is always constant as $T_o$. Therefore, $P_1(t_0)$ reduces to a constant matrix regardless of the value of $t_0$.

Let’s introduce new notations as follows:

$$
\tilde{P}_1 = P_1(t_0), \quad \tilde{W}_1 = W_1(t_0), \quad \tilde{W}_2(s) = W_2(t_0, s).
$$

Finally, the receding horizon predictive control is represented as a distributed state feedback strategy as follows:

$$
u(t_i) = \begin{cases} 
-B^T \tilde{P}_1 x(t) + \int_{-h}^{0} \tilde{P}_1 x(t + s) ds \\
\text{for } t_f - h < T_o
\end{cases}
$$

$$
-B^T \tilde{W}_1 x(t) + \int_{-h}^{0} \tilde{W}_2 x(t + s) ds \\
\text{for } 0 < T_o < t_f - h
$$

It is noted that the feedback strategy needs only the state trajectories for time interval $[t-h, t]$ and is invariant with time.

**Remark 2:** In order to solve partial differential equations given above, we utilize a numerical procedure introduced in [10]. The main idea of the method in [10] is that the original partial differential equations can be transformed into ordinary differential equations by appropriate change of variables.

Now we have constructed a receding horizon predictive control from the solution to a finite horizon dynamic game problem. However, the only thing we can say about the control is that it is obtained based on the receding horizon strategy. Nothing can be said about the asymptotic stability and $H_\infty$-norm boundedness. We therefore will investigate those issues in the next section.
3. Nonincreasing Monotonicity of a Saddle Point Value

Theorem 1: Given $\gamma > 0$, assume that there exist $X > 0$, $S$, $Y$, and $Y^T$ such that
\[
\begin{bmatrix}
(1, 1) & A_1 S + B Y_1 & B w & \mathbb{X} Q \mathbb{I} & Y^T X \\
* & -S & 0 & 0 & 0 \\
* & * & -\gamma^2 I & 0 & 0 \\
* & * & * & -I & 0 \\
* & * & * & * & -S
\end{bmatrix}
\leq 0.
\] (16)

where
\[(1, 1) = (A X + B Y) + (A X + B Y)^T.\]

If we choose terminal weighting matrices $F_1$ and $F_2$ such that $F_1 = X^{-1} F_2$ and $F_2 = S^{-1}$, the saddle point value $J^*(x_{t_0}, t_0, \sigma)$ satisfies the following nonincreasing monotonicity property:
\[
\frac{\partial J^*(x_{t_0}, t_0, \sigma)}{\partial \sigma} \leq 0, \quad \forall \sigma > t_0.
\] (17)

Proof:
\[
\frac{\partial J^*(x_{t_0}, t_0, \sigma)}{\partial \sigma} = \frac{1}{\Delta} \left\{ \int_{t_0}^{\sigma + \Delta} \left[ \mathbb{E}^T(\tau) Q \mathbb{E}(\tau) + \mathbb{E}^T(\tau) \mathcal{F} \mathbb{E}(\tau) \right] d\tau - \gamma^2 \mathbb{E}^T(\tau, x_{t_0}) \mathcal{F}^T(\tau, x_{t_0}) d\tau \right. \\
+ \mathbb{E}^T(\sigma + \Delta) \mathcal{F}_1 \mathbb{E}(\sigma + \Delta) + \int_{\sigma + \Delta}^{\text{horizon}} \mathbb{E}^T(\tau) \mathcal{F}_2 \mathbb{E}(\tau) d\tau \\
- \int_{\sigma - \Delta}^\sigma \left[ \mathbb{E}^T(\tau) Q \mathbb{E}(\tau) + \mathbb{E}^T(\tau) \mathcal{F} \mathbb{E}(\tau) \right] d\tau + \mathbb{E}^T(\sigma) \mathcal{F}_1 \mathbb{E}(\sigma)
\]
where the pair $(\mu, \nu)$ is a saddle point solution for $J(x_{t_0}, t_0, \sigma + \Delta, u, w)$ and the pair $(\bar{\mu}, \bar{\nu})$ is for $J(x_{t_0}, t_0, \sigma, u, v)$. \(\mathbb{E}\) denotes the state trajectory resulting from the strategies \(\bar{\mu}\) and \(\bar{\nu}\) that \(\nu\) denotes the state trajectory resulting from the strategies \(\mu\) and \(\nu\). Replace the strategy \(\bar{\mu}\) by \(\bar{\mu}\) and \(\bar{\nu}\) up to \(\sigma\) and use \(u(\tau) = K x(\tau) + K_1 x(\tau - h)\) and \(w(\tau) = \nu(\tau, x_{t_0})\) for \(\tau \geq \sigma\). It is noted that, since we have changed strategies, the resulting state trajectory is neither \(\hat{\mathbb{E}}\) nor \(\bar{\mathbb{E}}\). Let’s denote the resulting state trajectory by \(x\). Then we have
\[
\frac{\partial J^*(x_{t_0}, t_0, \sigma)}{\partial \sigma} \leq \lim_{\Delta \to 0} \frac{1}{\Delta} \left\{ \int_{\sigma - \Delta}^{\sigma + \Delta} \left[ \mathbb{E}^T(\tau) Q \mathbb{E}(\tau) + \mathbb{E}^T(\tau) \mathcal{F} \mathbb{E}(\tau) \right] d\tau - \gamma^2 \mathbb{E}^T(\tau, x_{t_0}) \mathcal{F}^T(\tau, x_{t_0}) d\tau \right. \\
+ \mathbb{E}^T(\sigma + \Delta) \mathcal{F}_1 \mathbb{E}(\sigma + \Delta) + \int_{\sigma + \Delta}^{\text{horizon}} \mathbb{E}^T(\tau) \mathcal{F}_2 \mathbb{E}(\tau) d\tau \\
- \int_{\sigma - \Delta}^\sigma \left[ \mathbb{E}^T(\tau) Q \mathbb{E}(\tau) + \mathbb{E}^T(\tau) \mathcal{F} \mathbb{E}(\tau) \right] d\tau + \mathbb{E}^T(\sigma) \mathcal{F}_1 \mathbb{E}(\sigma)
\]
\[
= \mathbb{E}^T(\sigma) \mathbb{Q} \mathbb{E}(\sigma) + \left[ K x(\sigma) + K_1 x(\sigma - h) \right]^T \\
\times \left[ K x(\sigma) + K_1 x(\sigma - h) \right] \\
\times [K x(\sigma) + K_1 x(\sigma - h)]^T \\
- \gamma^2 \mathbb{E}^T(\tau) \mathcal{F} \mathbb{E}(\tau) d\tau \right. \\
\times [K x(\sigma) + K_1 x(\sigma - h)]^T \\
- \gamma^2 \mathbb{E}^T(\tau) \mathcal{F} \mathbb{E}(\tau) d\tau \right)
\]
\[
- \gamma^2 \mathbb{E}^T(\tau) \mathcal{F} \mathbb{E}(\tau) d\tau \right)
\]
\[
= \mathbb{E}^T(\sigma) \mathbb{Q} \mathbb{E}(\sigma) + \left[ K x(\sigma) + K_1 x(\sigma - h) \right]^T \\
\times \left[ K x(\sigma) + K_1 x(\sigma - h) \right] \\
\times [K x(\sigma) + K_1 x(\sigma - h)]^T \\
- \gamma^2 \mathbb{E}^T(\tau) \mathcal{F} \mathbb{E}(\tau) d\tau \right.
\]
After substituting \(\hat{x}(\sigma) = (A + BK_1) x(\sigma) + (A_1 + BK_1) \hat{x}(\sigma - h) + B w(\sigma)\) into the above, we obtain
\[
\frac{\partial J^*(x_{t_0}, t_0, \sigma)}{\partial \sigma} \leq 0, \quad \forall \sigma > t_0.
\] (17)

where
\[
\Lambda_{11} = (A + BK_1)^T F_1 + F_1 (A + BK) + Q + K^T K + F_2
\]
\[
\Lambda_{12} = F_1 (A_1 + BK_1) + K^T K_1.
\]

It is apparent that, if $\Lambda \leq 0$, nonincreasing monotonicity in (17) holds. $\Lambda \leq 0$ can be rewritten as follows:
\[
\Lambda_{11} = (A + BK_1)^T F_1 + F_1 (A + BK) + Q + K^T K + F_2
\]
\[
\Lambda_{12} = F_1 (A_1 + BK_1) + K^T K_1.
\]

Pre- and post-multiply the above matrix inequality by $\text{diag}[F_1^{-1}, F_2^{-1}]$. The above inequality is then equivalently changed into (16) according to the Schur complement. This completes the proof.

Remark 3: The nonincreasing monotonicity of the saddle point value implies that the saddle point value does not increase even though we increase the horizon length. As will be shown in the next section, this property plays an important role in the RHC’s achieving closed-loop stability and $H_\infty$-norm boundedness.

4. Asymptotic Stability and $H_\infty$-norm Boundedness

In this section, we show that the proposed receding horizon predictive control achieves the closed-loop asymptotic stability for zero disturbance and satisfies the $H_\infty$-norm boundedness for zero initial condition.

Theorem 2: Given $Q > 0$ and $\gamma > 0$, if $\frac{\partial J^*(x_{t_0}, t_0, \sigma)}{\partial \sigma} \leq 0$ for $\sigma > t_0$, the system (1) controlled by the RHHPC is
asymptotically stable for zero disturbance and satisfies infinite horizon $H_\infty$-norm bound for zero initial condition.

Proof:

\[ J^*(x_1, t, t + T_p) \]
\[ = \int_t^{t+\theta} \left[ x^T(\tau)Qx(\tau) + \mu^T(\tau, x_\tau)\mu(\tau, x_\tau) \right] d\tau + J^*(x_{1+\theta}, t + \theta, t + T_p) \]

where the pair $(\mu^*, \nu^*)$ is a saddle point solution for $J(x_1, t, t + T_p, u, w)$. Replace the saddle point strategy $\nu^*(\tau, x_\tau)$ by $\nu(\tau, x_\tau)$ given by

\[ \nu(\tau, x_\tau) = \begin{cases} w(\tau) & , t \leq \tau < t + \theta \\ \nu^*(\tau, x_\tau) & , t + \theta \leq \tau \leq t + T_p \end{cases} \]

where $w(\tau)$ denotes an arbitrary signal. Recalling $J^*(x_1, t, t + T_p) = J(x_1, t, t + T_p, \mu^*, \nu^*)$, we obtain

\[ J^*(x_1, t, t + T_p) \leq J(x_1, t, t + T_p, \mu^*, \nu^*) \]

\[ = \int_t^{t+\theta} \left[ x^T(\tau)Qx(\tau) + \mu^T(\tau, x_\tau)\mu(\tau, x_\tau) - \gamma^2 w^T(\tau)w(\tau) \right] d\tau \]

\[ + J(\bar{x}_{1+\theta}, t + \theta, t + T_p, \mu^*, \nu^*) \]

where $\bar{x}$ denotes the state trajectory resulting from the strategy $\mu^*$ and $\nu$. It is noted that the strategy pair $(\mu^*, \nu^*)$ is the saddle point solution not only for $J(x_1, t, t + T_p, u, w)$ but also for $J(x_{1+\theta}, t + \theta, t + T_p, u, w)$ because the terminal time is $t + T_p$ for both cases. Therefore $J(\bar{x}_{1+\theta}, t + \theta, t + T_p, \mu^*, \nu^*) = J^*(\bar{x}_{1+\theta}, t + \theta, t + T_p)$. Furthermore, from the fact that $J^*(x_{1+\theta}, t + \theta, t + T_p) \leq 0$ for $\sigma > \theta$, it follows $J^*(\bar{x}_{1+\theta}, t + \theta, t + T_p) \geq J^*(\bar{x}_{1+\theta}, t + \theta, t + T_p + \theta)$. This, in turn, leads to

\[ J^*(\bar{x}_{1+\theta}, t + \theta, t + T_p + \theta) \geq J^*(x_{1+\theta}, t + \theta, t + T_p + \theta) \]

Therefore we obtain

\[ J^*(\bar{x}_{1+\theta}, t + \theta, t + T_p + \theta) - J^*(x_{1+\theta}, t + \theta, t + T_p) \]

\[ \leq \frac{1}{\theta} \int_t^{t+\theta} \left[ x^T(\tau)Qx(\tau) + \mu^T(\tau, x_\tau)\mu(\tau, x_\tau) - \gamma^2 w^T(\tau)w(\tau) \right] d\tau \]

When $\theta \to 0$, we have

\[ \frac{dJ^*(x_1, t, t + T_p)}{dt} \]

\[ \leq -[x^T(t)Qx(t) + u^T_R(x_1)u_R(x_1) - \gamma^2 w^T(t)w(t)] \]  

\[ = -\psi^T \left[ Q + \hat{P}_1B^TB^T \hat{P}_1 \hat{P}_1B \right] \psi - \gamma^2 w^T(t)w(t) \]

where

\[ \psi = \begin{bmatrix} x(t) \\ B^T \int_0^t \hat{P}_1(s)x(t + s)ds \end{bmatrix} \]

In case of $w(t) = 0$, we know that

\[ \frac{dJ^*(t, t + T_p)}{dt} \leq -\psi^T \left[ Q + \hat{P}_1B^TB^T \hat{P}_1 \hat{P}_1B \right] \psi \]

\[ \leq -x^T(t)Qx(t) \]

\[ \leq -\lambda_{\min}(Q)\|x(t)\|^2 \]

We can conclude that $J^*(x_1, t, t + T_p)$ is a Lyapunov-Krasovskii functional from the above derivation. Therefore, the closed-loop system is asymptotically stable. We then prove that $H_\infty$-norm bound is guaranteed for the closed-loop system. Consider an infinite horizon cost $J_w$ given by

\[ J_w = \int_0^\infty [x^T(t)Qx(t) + u^T_R(x_1)u_R(x_1) - \gamma^2 w^T(t)w(t)]dt \]

We have only to show that $J_w \leq 0$ for the proof of $H_\infty$-norm bound.

\[ J_w = \int_0^\infty \left[ x^T(t)Qx(t) + u^T_R(x_1)u_R(x_1) - \gamma^2 w^T(t)w(t) \right] dt \]

\[ + \frac{d}{dt}J^*(x_1, t, t + T_p)dt \]

\[ + J^*(x_0, 0, T_p) - J^*(x_1, t, t + T_p) \]

For zero initial condition, i.e. $x(s) = 0, s \in [-h, 0]$, $J^*(x_0, 0, T_p)$ is equal to zero. Furthermore, since the saddle point value is nonnegative-definite, we have $J^*(x_1, t, t + T_p) \geq 0$. This, in turn, leads to

\[ J_w \leq 0 \]

\[ \frac{d}{dt}J^*(x_1, t, t + T_p) \]

The integrand above is less than or equal to zero from (18). Consequently, we can conclude that $J_w \leq 0$. This completes the proof.

Theorem 2 states that the nonincreasing monotonicity of the saddle point value is the sufficient condition for the closed-loop stability and the $H_\infty$-norm boundedness. An LMI condition on the terminal weighting matrices under which the saddle point satisfies nonincreasing boundedness was given in Theorem 1. Therefore, we can construct the following corollary:

Corollary 2: Given $Q > 0$ and $\gamma > 0$, if the LMI (16) is feasible and we can obtain two terminal weighting matrices $F_1$ and $F_2$, the system (1) controlled by the proposed RHHPF is asymptotically stable for zero disturbance and satisfies infinite horizon $H_\infty$ norm bound for zero initial condition.

5. Numerical Example

In this section, we provide a numerical example in order to illustrate the properties of the proposed RHHPF. Consider a chemical reactor system taken from [11]. The system matrices are given by

\[ A = \begin{bmatrix} -4.93 & -1.01 & 0 & 0 \\ -3.20 & -5.30 & -12.8 & 0 \\ -6.40 & 0.347 & -32.5 & -1.04 \\ 0 & 0.833 & 11.0 & -3.96 \end{bmatrix} \]
problem. The terminal weighting term is parameterized by two terminal weighting matrices. Secondly, we derived a saddle point solution to a finite horizon dynamic game problem using the proposed cost function. Thirdly, a receding horizon predictive control was constructed from the obtained solution. We showed that, under the nonincreasing monotonicity condition of a saddle point value, the proposed receding horizon predictive control is stabilizing and satisfies the $H_{\infty}$-norm bound. We proposed an LMI condition on the terminal weighting matrices, under which the saddle point value satisfies the nonincreasing monotonicity. Main work of this paper is the development of the RHHPC for time-delay systems.

References


6. Conclusions

In this paper, we proposed the receding horizon $H_{\infty}$ predictive control (RHHPC) for linear systems with a state-delay. Firstly, we proposed a new cost function for a dynamic game

$$A_1 = \text{diag} \{1.92, 1.92, 1.87, 0.724\}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B_w = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

We chose $Q = I$. For $\gamma = 0.85$, we obtained terminal weighting matrices satisfying the monotonicity condition in (16) as follows:

$$F_1 = \begin{bmatrix} 1.8193 & -0.6531 & 0.2726 & -0.0417 \\ -0.6531 & 0.3979 & -0.1397 & 0.0605 \\ 0.2726 & -0.1397 & 0.1042 & 0.0467 \\ -0.0417 & 0.0605 & 0.0467 & 0.2169 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} 3.2895 & -1.3429 & 0.5725 & 0.0081 \\ -1.3429 & 0.6921 & -0.2584 & 0.0258 \\ 0.5725 & -0.2584 & 0.1380 & 0.0301 \\ 0.0081 & 0.0258 & 0.0301 & 0.0466 \end{bmatrix}$$

In order to illustrate the stability and the $H_{\infty}$-norm boundedness, we applied the disturbance input $w(t)$ that has the shape in Figure 1. Figure 2 shows the state response of the closed-loop system to the disturbance in Figure 1. It clearly shows that the resulting closed-loop system is stable. The value of $\|z\|_{\infty}/\|w\|_{2}$ was computed to be 0.5384, which is less than $\gamma = 0.85$. This supports that the closed-loop system satisfies $H_{\infty}$-norm boundedness.