A Comparative Study of Different Reliability Calculation Algorithms

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Abstract – In this paper, three reliability calculation algorithms: Monte Carlo Simulation (MCS), Reliability Index Approach (RIA), and Sensitivity-based Monte Carlo Simulation (SMCS) are studied. Their efficiency and accuracy are validated by analytic test functions.

1. Introduction

In the engineering applications, the uncertainty in design variables is inevitable. Therefore, nowadays, the ability to accurately characterize and propagate these uncertainties is increasingly important. Usually, the reliability defined as the probability of a design being in the feasible region to evaluate its quality.

1.1 Reliability Index Approach

For a given design problem, the constraint function can be calculated as:

\[ g(X) = 0 \]

where \( g(X) \) is the performance function, and \( X \) is the design variable vector. For a linear design space, the shortest distance becomes an optimization problem:

\[ \min_{x \in X} \| R(x) \| \]

where \( R(x) \) is the residual vector in Galerkin's approximation; other symbols have their usual meanings in Finite Element Method (FEM). Once the sensitivity is obtained from (7), the reliability analysis can be performed to an approximated analytic function.

1.2 Monte Carlo Simulation

In the MCS, for the reliability of a design \( X \) with respect to the constraint \( g(X) = 0 \), \( N \) different samples are generated in a certain confidence interval \( |\mu_i - k\sigma_i, \mu_i + k\sigma_i| = 1.2 \cdots M \). If \( n \) samples satisfy the constraint \( g(X) = 0 \), then the reliability of the design \( X \) can be obtained as:

\[ R(g(X) > 0) = \frac{n}{N}. \]  (5)

1.3 Sensitivity-based Monte Carlo Simulation

Based on the requirement of reliability evaluation for real engineering application, the sensitivity-based MCS method is proposed. In SMCS, the sensitivity analysis is applied to calculate constraint function by constructing an approximated analytic function especially for the nonlinear performance constraints.

2. Reliability Calculation Methods

In order to simplify explanation, all the design variables are assumed as uncertain ones. For a design problem, the constraint function \( g(X) = 0 \) defines feasible region; \( g(X) > 0 \) defines infeasible region. The comparison will give some guidelines to reliability-based design optimization with the selection of proper reliability calculation method.

2.1 Reliability Index Approach

For a given design, the Hasofer-Lind reliability index \( \beta \) [1] is defined as follows:

\[ \beta = \frac{\sum_i u_i^2}{\sum_i u_i^2 \sigma_i^2} \]  (1)

where \( u_i = (x_i - \mu_i) / \sigma_i \).

2.2 Monte Carlo Simulation

In the MCS, for the reliability of a design \( X \) with respect to the constraint function \( g(X) = 0 \), \( N \) different samples are generated in a certain confidence interval \( |\mu_i - k\sigma_i, \mu_i + k\sigma_i| = 1.2 \cdots M \). If \( n \) samples satisfy the constraint \( g(X) = 0 \), then the reliability of the design \( X \) can be obtained as:

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2.3 Sensitivity-based Monte Carlo Simulation

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For a given determinant design \( X_0 \), the constraint functions in uncertain regions are approximated to linear ones using their gradient vectors as follows:

\[ g(X) = g(X_0) + \nabla g(X_0) \cdot (X - X_0). \]  (6)

The gradient vector is calculated as follows [2]:

\[ \nabla g(X_0) = \frac{\partial g}{\partial X_i} \bigg|_{X_0} = \frac{g|_{X_0}}{\| \nabla g \|_{X_0}} \frac{\partial R}{\partial B} \bigg|_{X_0} \]  (7)

where \( R \) is the residual vector in Galerkin's approximation; other symbols have their usual meanings in Finite Element Method (FEM). Once the sensitivity is obtained from (7), the reliability analysis can be performed to an approximated analytic function.

3. Results and Conclusions

The reliability analysis are applied to the following analytic constraint functions as shown in Fig.1:

\[ g_1(X) = \frac{(x_1 - 2.25)(x_2 - 1.3)}{5} \leq 1 \geq 0 \]  (8-1)
\[ g_2(X) = \frac{(x_1 + 2.47)^2 + 8x_2 + 5}{80} \leq 1 \geq 0 \]  (8-2)
\[ g_3(X) = -1.6(x_1 - 4) - x_2 + 4.16 \leq 0 \]  (8-3)

where the design variables \( x_1 \) and \( x_2 \) (0 \( x_1, x_2 \leq 10 \)) follows the same Gaussian distribution with standard deviation \( \sigma = 0.3 \). In the given design space, four testing points \( A(3.6, 3.5), B(3.75, 3.9), C(4.0, 4.16) \) and \( D(4.3, 4.5) \) are selected. In the MCS and SMCS, the confidence level of uncertain variable is set to 95%, and the number of MCS and SMCS trials is set to 2,000,000.
For \( g_1(x) \) and \( g_2(x) \), testing points (B, C, D) and (A, B, C) are selected, respectively. The comparison results are shown in Table 1, Fig. 2 and Table 2, Fig. 3, respectively. From the comparisons, it is obvious that the SMCS can achieve almost the same accuracy as MCS, while the accuracy of RIA is a little worse than MCS (Point C and D).

For the linear case, \( g_3(x) \), as shown in Fig. 4, the reliability obtained by MCS may coincide with the other two methods. As shown in Fig. 1, due to the difference in the feasible region, the reliability of design C for three limit state functions should be different. However, the RIA finds the same reliability index, as shown in Fig. 5, which is not accurate for the nonlinear limit functions. From the above analysis, we can get the following conclusions:

1. The reliability from MCS is usually taken as a benchmark with its highest accuracy if the trials are as big enough. However, it is much more time-consuming for engineering applications;
2. Once the MPPF is found, the RIA is the most efficient method to calculate reliability. There are following drawbacks: searching of MPPF is related with optimization problem; if the reliability analysis is combined with design optimization, it will be time-consuming; the accuracy is worse especially for nonlinear limit state functions;
3. Compared with MCS, the SMCS can save a lot of computational time with the help of sensitivity analysis; At the same time, the accuracy is no worse than the RIA.

We can demonstrate that, until now, the SMCS is the best choice for the engineering applications.

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**[Fig. 1] Limit state functions**

![Image of limit state functions]

**[Fig. 2] Reliability comparison of constraint 1**

![Image of reliability comparison]

**[Fig. 3] Reliability comparison of constraint 2**

![Image of reliability comparison]

**[Fig. 4] Reliability comparison of constraint 3**

![Image of reliability comparison]

**[Fig. 5] Reliability comparison of point C**

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**[Table 1] Reliability of constraint function 1**

<table>
<thead>
<tr>
<th>Method</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>MCS</td>
<td>9.145E-2</td>
<td>0.484294</td>
<td>0.947406</td>
</tr>
<tr>
<td>SMCS</td>
<td>8.486E-2</td>
<td>0.486787</td>
<td>0.940090</td>
</tr>
<tr>
<td>RIA</td>
<td>7.288E-2</td>
<td>0.500000</td>
<td>0.935162</td>
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</tbody>
</table>

**[Table 2] Reliability of constraint function 2**

<table>
<thead>
<tr>
<th>Method</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>MCS</td>
<td>0.998078</td>
<td>0.896754</td>
<td>0.484294</td>
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<tr>
<td>SMCS</td>
<td>0.993423</td>
<td>0.885806</td>
<td>0.486787</td>
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<tr>
<td>RIA</td>
<td>0.995587</td>
<td>0.881464</td>
<td>0.500000</td>
</tr>
</tbody>
</table>

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**[Reference]**
