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EVALUATION OF THE AMOUNT OF MISSING INFORMATION

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1. Introduction

Let $(S, \boldsymbol{S})$ be a measurable state space, W be a prior probability measure on $(S, \boldsymbol{S})$. Let W be absolutely continuous with respect to a measure λ on $(S, \boldsymbol{S})$ and let $dW=w(s)d\lambda$.

Then the uncertainty measure $H(W)$ of the unknown state $s\in S$ is defined by

$$H(W)=-\int w(s)\log w(s)\,d\lambda. \qquad (1.1)$$

Specifically, if $S=\{s_1, s_2\cdots, s_n\}$ and $W=(w(s_1), w(s_2), \cdots, w(s_n))$ $\equiv(w_1, w_2, \cdots, w_n)$, where $w_i\geq 0$, $\sum_{i=1}^{n} w_i=1$, then

$$H(W)=-\sum_{i=1}^{n} w_i\log w_i \text{ [1]}. \qquad (1.2)$$

In [4], the amount of missing information is defined as follows:

Let us denote by $X(m)$ the random m dimensional vector $(X_1, X_2, \cdots, X_m)$. Let us suppose that the distribution of $X(m)$ depends on a state S, which may take on n different values $s_1, s_2, \cdots, s_n$. Let $X(m)$ be a random variable on a measurable space $(X(m), \boldsymbol{X}(m))$ whose probability density function $f(x(n)|s)$ with respect to a measure μ on $(X(m), \boldsymbol{X}(m))$ is assumed to be known.

We suppose that the random variables X_i $(i=1, \cdots, m)$ are independent under the condition that $S=s_i$ $(i=1, \cdots, n)$. Let $f_1(x(m))$, $f_2(x(m))$, $\cdots$ and $f_n(x(m))$ denote the density functions for $S=s_1$, $S=s_2, \cdots$ and $S=s_n$ respectively.

The amount of information contained in $X(m)$ concerning S is defined as follow:

$$I_m=H(W)-E[H(W(X(m)))] \qquad (1.3)$$

where $H(W(X(m)))$ is conditional entropy of S given $X(m)$, that is

$$H(W(X(m)))=\sum_{i=1}^{n} P_r(S=s_i|X(m))\log\frac{1}{P_r(S=s_i|X(m))}. \qquad (1.4)$$

and $E[H(W(X(m)))]$ denotes the expectation of the random variable $H(W(X(m)))$.
The quantity $E[H(W(X(m)))]$ is interpreted as *the amount of missing information on S after observing $X(m)$.*

A. Rényi, established the following inequality between the amount of missing information and the error of the standard decision in case $S=\{s_1, s_2\}$, that is $\frac{1}{2}\varepsilon_m{}^2 \leq E\{H_2(W(X(m)))\} \leq h(\varepsilon_m^2)$ [5]. (1.5)

In this paper, we shall generalize Rényi's result in the lower inequality and obtain the upper bound in the case $S=\{s_1, s_2, \cdots, s_n\}$.

In section 2, the standard decision in case $S=\{s_1, s_2, \cdots, s_n\}$ will be considered and the amount of missing information will be compared with the error of standard decision.

In section 3, an upper bound for the amount of missing information will be given by means of the error function and the success rate function.

2. Evaluation of the amount of missing information by means of the error of the standard decision

Let us introduce the following decision rule. The most natural decision after having observed $X(m)=x(m)$ is essentially the same as the reason applied in case $S=\{s_1, s_2\}$ [5].

We assume that s_i is selected such that $w_i f(x(m))$ is the greatest among $w_1 f_1(x(m))$, $w_2 f_2(x(m))$, $\cdots$ or $w_n f_n(x(m))$, that is, s_i is the greatest posterior probability, and if $w_1 f_1(x(m)) = w_2 f_2(x(m)) = \cdots = w_n f_n(x(m))$, one make a random choice among $s_1, s_2, \cdots$or s_n with probabilities $w_1, w_2, \cdots$ or w_n respectively. We shall call this the *standard decision.*

Let us define random variable $\Phi_m = \Phi_m(x(m))$ as follow:

$$\Phi_m = s_i \text{ if the standard decision means acceptance of } s_i, \quad i=1, 2, \cdots, n. \tag{2.1}$$

The error ε_m^n of the standard decision after taking m observation is defined as the probability of the standard decision being false, and ε_m^n is expressed as

$$\begin{aligned}\varepsilon_m^n = P_r(\Phi_m \neq S) = w_1[&P_r(\Phi_m = s_2 \mid S=s_1) + P_r(\Phi_m = s_3 \mid S=s_1) + \cdots \\ &+ P_r(\Phi_m = s_n \mid S=s_1)] \\ &+ w_2[P_r(\Phi_m = s_1 \mid S=s_2) + P_r(\Phi_m = s_3 \mid S=s_2) + \cdots \\ &+ P_r(\Phi_m = s_n \mid S=s_2)]\end{aligned} \tag{2.2}$$

$$+\cdots\cdots$$
$$+w_m[P_r(\Phi_m=s_1|S=s_n)+P_r(\Phi_m=s_2|S=s_n)+\cdots$$
$$+P_r(\Phi_m=s_{n-1}|S=s_n)].$$

In a decision problem which is available to a decision maker, we devide the sample space R^m into the disjoint acceptance regions, $R^m_{(1)}, R^m_{(2)}, \cdots, R^m_{(n)}$ such that $\Phi_m=s_j$ is accepted when $x(m)\in R^m_{(j)}$, $j=1, 2, \cdots, n$.

With this specification we have:

$$\begin{aligned}\varepsilon^n_m=\Big\{&w_1\Big[\int_{R^m_{(2)}}f_1(x(m))dx(m)+\int_{R^m_{(3)}}f_1(x(m))dx(m)+\cdots\\&+\int_{R^m_{(n)}}f_1(x(m))dx(m)\Big]+w_2\Big[\int_{R^m_{(1)}}f_2(x(m))dx(m)\\&+\int_{R^m_{(3)}}f_2(x(m))dx(m)+\cdots+\int_{R^m_{(n)}}f_2(x(m))dx(m)\Big]\\&+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\&+w_n\Big[\int_{R^m_{(1)}}f_n(x(m))dx(m)+\int_{R^m_{(2)}}f_n(x(m))dx(m)+\cdots\\&+\int_{R^m_{(n-1)}}f_n(x(m))\,dx(m)\Big]\Big\}\end{aligned}\tag{2.3}$$

where $dx(m)$ stands for $dx_1dx_2\cdots dx_m$ and for $x(m)\in R^m_{(i)}$, it holds that

$$w_if_i(x(m))\geqq w_jf_j(x(m)).\tag{2.4}$$

We obtain the following theorem:

THEOREM 1.

$$\frac{\varepsilon^n_m}{n}\leqq E[H_2(W(X(m)))]\tag{2.5}$$

where index 2 denotes logarithm with base 2.

Proof. For simplicity, we shall denote $f_1(x(m)), f_2(x(m)), \cdots, f_n(x(m))$ I_i, and J as follows:

$$f_1\equiv f_1(x(m))=\prod_{r=1}^{m}f_1(x_r),\quad f_2\equiv f_2(x(m))=\prod_{r=1}^{m}f_2(x_r),\cdots,\ f_n\equiv f_n(x(m))$$
$$=\prod_{r=1}^{m}f_n(x_r),\ f\equiv f(x(m))=w_1f_1+w_2f_2+\cdots+w_nf_n,$$
$$I_i\equiv\frac{w_if_i}{f}\log\frac{f}{w_if_i},\ i=1, 2, \cdots, n,\tag{2.6}$$
and $J\equiv\sum_{i=1}^{n}I_i$.

One has clearly

$$E\{H_2(W(X(m)))\}=\int_{R^m}H_2(W(X(m)))fdx(m)$$

$$=\int_{R^m} H_2(W(X(m)))w_1f_1dx(m)+\cdots+\int_{R^m} H_2(W(X(m)))w_nf_ndx(m)$$

$$=\int_{R^m}[J]w_1f_1dx(m)+\cdots+\int_{R^m}[J]w_nf_ndx(m)$$

$$\geqslant\int_{R^m}[J-I_1]w_1f_1dx(m)+\cdots+\int_{R^m}[J-I_n]w_nf_ndx(m)$$

$$\geqslant\int_{R^m_{(1)}}[J-I_1]w_1f_1dx(m)+\cdots+\int_{R^m_{(n)}}[J-I_n]w_nf_ndx(m). \qquad (2.7)$$

Let us consider the first term of r.h.s. of the inequality (2.7), we notice that

$$\int_{R^m_{(1)}}[J-I_1]w_1f_1dx(m)=\int_{R^m_{(1)}}\frac{w_1f_1}{f}\log\left(1+\frac{w_1f_1}{w_2f_2}+\cdots+\frac{w_nf_n}{w_2f_2}\right)w_2f_2dx(m)$$

$$+\int_{R^m_{(1)}}\frac{w_1f_1}{f}\log\left(1+\frac{w_1f_1}{w_3f_3}+\cdots+\frac{w_nf_n}{w_3f_3}\right)w_3f_3dx(m)$$

$$+\cdots\cdots\cdots$$

$$+\int_{R^m_{(1)}}\frac{w_1f_1}{f}\log\left(1+\frac{w_1f_1}{w_nf_n}+\cdots+\frac{w_{n-1}f_{n-1}}{w_nf_n}\right)w_nf_ndx(m).$$

By the inequality (2.4), it follows that

$$\int_{R^m_{(1)}}[J-I_1]w_1f_1dx(m)\geqslant\frac{1}{n}\int_{R^m_{(1)}}(w_2f_2+w_3f_3+\cdots+w_nf_n)dx(m). \qquad (2.8)$$

Similary, we obtain:

$$\int_{R^m_{(2)}}[J-I_2]w_2f_2dx(m)\geqslant\frac{1}{n}\int_{R^m_{(2)}}(w_1f_1+w_3f_3+\cdots+w_nf_n)dx(m), \qquad (2.9)$$

$$\cdots\cdots\cdots$$

$$\int_{R^m_{(n)}}[J-I_n]w_nf_ndx(m)\geqslant\frac{1}{n}\int_{R^m_{(n)}}(w_1f_1+w_2f_2+\cdots+w_{n-1}f_{n-1})dx(m). \qquad (2.10)$$

By the inequalities (2.8), (2.9) and (2.10), we have

$$E\{H_2(W(X(m)))\}\geqslant\frac{1}{n}\varepsilon^n_m.$$

Thus the proof is completed.

3. The upper bound for the amount of missing information

DEFINITION 1. The rate of success is defined by

$$\varphi=\int_{R^m_{(1)}}w_1f_1dx(m)+\int_{R^m_{(2)}}w_2f_2dx(m)+\cdots+\int_{R^m_{(n)}}w_nf_ndx(m). \qquad (3.1)$$

DEFINITION 2. The expected logarithm success rate is defined by

$$\varphi^*=\sum_{i=1}^{n}\int_{R^m_{(i)}}(\log w_if_i)fdx(m). \qquad (3.2)$$

LEMMA 1. *One has*

$$\int_{R^m}\left[H\left(\frac{f-w_1f_1}{f},\ \frac{f-w_2f_2}{f},\ \cdots,\ \frac{f-w_nf_n}{f}\right)\right]fdx(m)$$

$$\geqslant(n-1)\{E(\log f)-\log(n-1)-\varphi^*\}. \tag{3.3}$$

Proof. By the definition of $H(\cdot)$, we have

$$\int_{R^m}\left[H\left(\frac{f-w_1f_1}{f},\ \frac{f-w_2f_2}{f},\ \cdots,\ \frac{f-w_nf_n}{f}\right)\right]fdx(m)$$

$$=\int_{R^m}\left[\sum_{i=1}^{n}\left(\frac{f-w_if_i}{f}\log\frac{f}{f-w_if_i}\right)\right]fdx(m). \tag{3.4}$$

For the sake of brevity, let us denote $R=\sum_{i=1}^{n}\left(\frac{f-w_if_i}{f}\log\frac{f}{f-w_if_i}\right)$.

Then, the equation (3.4) becomes

$$\int_{R^m}[R]fdx(m)=\int_{R^m_{(1)}}[R]fdx(m)+\int_{R^m_{(2)}}[R]fdx(m)+\cdots+\int_{R^m_{(n)}}[R]fdx(m). \tag{3.5}$$

The first term of the r. h. s of the equation (3.5) is

$$\int_{R^m_{(1)}}[R]fdx(m)\geqslant\int_{R^m_{(1)}}\left[\frac{f-w_1f_1}{f}\log\frac{f}{(n-1)w_1f_1}\right.$$

$$\left.+\frac{f-w_2f_2}{f}\log\frac{f}{(n-1)w_1f_1}+\cdots+\frac{f-w_nf_n}{f}\log\frac{f}{(n-1)w_1f_1}\right]fdx(m)$$

$$=\int_{R^m_{(1)}}\left[\left(\frac{f-w_1f_1}{f}+\frac{f-w_2f_2}{f}+\cdots+\frac{f-w_nf_n}{f}\right)\log\frac{f}{(n-1)w_1f_1}\right]fdx(m)$$

$$=\int_{R^m_{(1)}}\left\{(n-1)\log\frac{f}{(n-1)w_1f_1}\right\}fdx(m) \tag{3.6}$$

Similary,

$$\int_{R^m_{(2)}}[R]fdx(m)\geqslant\int_{R^m_{(2)}}\left\{(n-1)\log\frac{f}{(n-1)w_2f_2}\right\}fdx(m). \tag{3.7}$$

....

$$\int_{R^m_{(n)}}[R]fdx(m)\geqslant\int_{R^m_{(n)}}\left\{(n-1)\log\frac{f}{(n-1)w_nf_n}\right\}fdx(m). \tag{3.8}$$

Therefore, we have

$$\int_{R^m}[R]fdx(m)\geqslant(n-1)\{E(\log f)-\log(n-1)-\varphi^*\}.$$

LEMMA 2. *For every nonnegative function $K(x)$ and every concave function $h(x)$, we have the following integral form of Jensen's inequality*

$$\frac{\int_A h(a(x))k(x)dx}{\int_A k(x)dx}\leqslant h\left(\frac{\int_A a(x)k(x)dx}{\int_A k(x)dx}\right). \tag{3.9}$$

Proof. see [2] or [4].

THEOREM 2. *One has*

$$E\{H_2(W(X(m)))\} \leqq h(\varepsilon_m^n + \sum_{l=1}^{n-1} h(\varphi + K(l) - (n-1)\{E(\log f) - \log(n-1) - \varphi^*\}. \tag{3.10}$$

where

$$h(x) = \begin{cases} x\log\frac{1}{x} + (1-x)\log\frac{1}{(1-x)}, & 0<x<1. \\ 0 & x=0 \text{ or } 1. \end{cases} \tag{3.11}$$

$$K(l) = \begin{cases} \sum_{i=1}^{l}\left(\beta_{il+1} - \int_{R_{(i)}^m} w_i f_i dx(m)\right) & \text{if } i \leqq l \\ & l=1,2,\cdots,n-1 \\ \sum_{i=l+1}^{n}\left(\beta_{il} - \int_{R_{(i)}^m} w_i f_i dx(m)\right) & \text{if } i>l, \end{cases} \tag{3.12}$$

$$\beta_{ij} = \int_{R_{(i)}^m} (f - w_j f_j) dx(m), \quad i, j = 1, 2, \cdots, n. \tag{3.13}$$

Proof. We have

$$E\{H_2(W(X(m)))\} = \int_{R^m}\left[\sum_{i=1}^{n}\left(\frac{w_i f_i}{f}\log\frac{f}{w_i f_i}\right)\right] f dx(m)$$

$$= \int_{R^m}\left[\sum_{i=1}^{n} h\left(\frac{w_i f_i}{f}\right) f dx(m) - \int_{R^m}\sum_{i=1}^{n}\left(\frac{f - w_i f_i}{f}\log\frac{f}{f - w_i f_i}\right)\right] f dx(m). \tag{3.14}$$

The first term of the r. h. s of the equation (3.14) is

$$\int_{R^m}\left[\sum_{i=1}^{n} h\left(\frac{w_i f_i}{f}\right)\right] f dx(m) = \int_{R_{(1)}^m}\left[\sum_{i=1}^{n} h\left(\frac{w_i f_i}{f}\right)\right] f dx(m) + \cdots$$

$$+ \int_{R_{(n)}^m}\left[\sum_{i=1}^{n} h\left(\frac{w_i f_i}{f}\right)\right] f dx(m).$$

Since $h(x) = h(1-x)$, we also have

$$\int_{R^m}\left[\sum_{i=1}^{n} h\left(\frac{w_i f_i}{f}\right)\right] f dx(m)$$

$$= \int_{R^m}\left[\sum_{i=1}^{n} h\left(\frac{f - w_i f_i}{f}\right)\right] f dx(m)$$

$$= \int_{R_{(1)}^m}\left[\sum_{i=1}^{n} h\left(\frac{f - w_i f_i}{f}\right)\right] f dx(m) + \int_{R_{(2)}^m}\left[\sum_{i=1}^{n} h\left(\frac{f - w_i f_i}{f}\right)\right] f dx(m)$$

$$+ \cdots + \int_{R_{(n)}^m}\left[\sum_{i=1}^{n} h\left(\frac{f - w_i f_i}{f}\right)\right] f dx(m). \tag{3.15}$$

By the Lemma 2, we have

$$\int_{R_{(1)}^{m}}\left[\sum_{i=1}^{n}h\left(\frac{f-w_{i}f_{i}}{f}\right)\right]fdx(m)\leqq\int_{R_{(1)}^{m}}fdx(m)\sum_{i=1}^{n}h\left(\frac{\int_{R_{(1)}^{m}}\left(\frac{f-w_{i}f_{i}}{f}\right)fdx(m)}{\int_{R_{(1)}^{m}}fdx(m)}\right)$$

$$=\int_{R_{(1)}^{m}}fdx(m)\left[\sum_{i=1}^{n}h\left(\frac{\int_{R_{(1)}^{m}}(f-w_{i}f_{i})fdx(m)}{\int_{R_{(1)}^{m}}fdx(m)}\right)\right] \tag{3.16}$$

Similary,

$$\int_{R_{(2)}^{m}}\left[\sum_{i=1}^{n}h\left(\frac{f-w_{i}f_{i}}{f}\right)\right]fdx(m)\leqq\int_{R_{(2)}^{m}}fdx(m)\left[\sum_{i=1}^{n}h\left(\frac{\int_{R_{(2)}^{m}}(f-w_{i}f_{i})fdx(m)}{\int_{R_{(2)}^{m}}fdx(m)}\right)\right], \tag{3.17}$$

....

$$\int_{R_{(n)}^{m}}\left[\sum_{i=1}^{n}h\left(\frac{f-w_{i}f_{i}}{f}\right)\right]fdx(m)\leqq\int_{R_{(n)}^{m}}fdx(m)\left[\sum_{i=1}^{n}h\left(\frac{\int_{R_{(n)}^{m}}(f-w_{i}f_{i})fdx(m)}{\int_{R_{(n)}^{m}}fdx(m)}\right)\right]. \tag{3.18}$$

For simplicity, we shall set $\alpha_{i}=\int_{R_{(i)}^{m}}fdx(m)$.
Therefore, we have

$$\int_{R^{m}}\left[\sum_{i=1}^{n}h\left(\frac{f-w_{i}f_{i}}{f}\right)\right]fdx(m)$$

$$\leqq\alpha_{1}\left[\sum_{j=1}^{n}h\left(\frac{\beta_{1j}}{\alpha_{1}}\right)\right]+\alpha_{2}\left[\sum_{j=1}^{n}h\left(\frac{\beta_{2j}}{\alpha_{2}}\right)\right]+\cdots+\alpha_{n}\left[\sum_{j=1}^{n}h\left(\frac{\beta_{nj}}{\alpha_{n}}\right)\right]$$

$$=\alpha_{1}h\left(\frac{\beta_{11}}{\alpha_{1}}\right)+\alpha_{2}h\left(\frac{\beta_{22}}{\alpha_{2}}\right)+\cdots+\alpha_{n}h\left(\frac{\beta_{nn}}{\alpha_{n}}\right) \tag{3.19}$$

$$+\alpha_{1}h\left(\frac{\beta_{12}}{\alpha_{1}}\right)+\alpha_{2}h\left(\frac{\beta_{21}}{\alpha_{2}}\right)+\cdots+\alpha_{n}h\left(\frac{\beta_{n1}}{\alpha_{n}}\right) \tag{3.20}$$

$$+\cdots\cdots\cdots\cdots$$

$$+\alpha_{1}h\left(\frac{\beta_{1n}}{\alpha_{1}}\right)+\alpha_{2}h\left(\frac{\beta_{2n}}{\alpha_{2}}\right)+\cdots+\alpha_{n}h\left(\frac{\beta_{nn-1}}{\alpha_{n}}\right). \tag{3.21}$$

Since $\alpha_{1}+\alpha_{2}+\cdots+\alpha_{n}=\int_{R^{m}}fdx(m)=1$ and $h(x)$ is concave function, the quantity (3.19) is less than

$$h\left(\frac{\beta_{11}}{\alpha_{1}}\alpha_{1}+\frac{\beta_{22}}{\alpha_{2}}\alpha_{2}+\cdots+\frac{\beta_{nn}}{\alpha_{n}}\alpha_{n}\right)$$

$$=h(\beta_{11}+\beta_{22}+\cdots+\beta_{nn})$$

$$=h(\varepsilon_{m}^{n}). \tag{3.22}$$

Similary, the quantity (3.20) is less than

$$h(\beta_{12}+\beta_{21}+\cdots+\beta_{n1})$$
$$=h\Big[\Big(\int_{R_{(1)}^m} w_1 f_1 dx(m)+\int_{R_{(2)}^m} w_2 f_2 dx(m)+\cdots+\int_{R_{(n)}^m} w_n f_n dx(m)\Big)$$
$$+\Big(\beta_{12}-\int_{R_{(1)}^m} w_1 f_1 dx(m)+\beta_{21}\int_{R_{(2)}^m} w_2 f_2 dx(m)+\cdots+\beta_{n1}-\int_{R_{(n)}^m} w_n f_n dx(m)$$
$$=h(\varphi+K_{(1)})\Big). \tag{3.23}$$

and the quantity (3.21) is less than

$$h(\beta_{1n}+\beta_{2n}+\cdots+\beta_{nn-1})$$
$$=h\Big[\Big(\int_{R_{(1)}^m} w_1 f_1 dx(m)+\int_{R_{(2)}^m} w_2 f_2 dx(m)+\cdots+\int_{R_{(n)}^m} w_n f_n dx(m)\Big)$$
$$+\Big(\beta_{1n}-\int_{R_{(1)}^m} w_1 f_1 dx(m)+\beta_{2n}-\int_{R_{(2)}^m} w_2 f_2 dx(m)+\cdots+\beta_{nn-1}-\int_{R_{(n)}^m} w_n f_n dx(m)\Big)$$
$$=h(\varphi+K_{(n-1)}). \tag{3.24}$$

By the equations (3.22), (3.23) and (3.24), we have

$$\int_{R^m}\Big[\sum_{i=1}^n h\Big(\frac{f-w_i f_i}{f}\Big)\Big] f dx(m) \leqslant h(\varepsilon_m)+h(\varphi+K_{(1)})+\cdots+h(\varphi+K_{(n-1)})$$
$$=h(\varepsilon_m^n+\sum_{l=1}^{n-1} h(\varphi+K_{(l)}). \tag{3.25}$$

By the Lemma 1 and the equation (3.14), we obtain that

$$E\{H_2(W(X(m)))\} \leqslant h(\varepsilon_m^n)+\sum_{l=1}^{n-1} h(\varphi+K_{(l)})-(n-1)\{E(\log f)-\log(n-1)-\varphi^*\}. \tag{3.26}$$

From the results of theorem 1 and theorem 2, we can obtain the main theorem as follows:

THEOREM 3.

$$\frac{\varepsilon_m^n}{n} \leqslant E\{H_2(W(X(m)))\} \leqslant h(\varepsilon_m^n)+\sum_{l=1}^{n-1} h(\varphi+K_{(l)})-(n-1)\{E(\log f)$$
$$-\log(n-1)-\varphi^*\}$$

where $h(x)$, $\varepsilon_m^n, \varphi, \varphi^*$ and $K_{(l)}$ are the same functions defined in the above.

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