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ON THE W-SPACES

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The object of this paper is to generalize the usual W-spaces [1] and by using the dual function $\rho_i^*(\eta_i)$ for a given function $\rho_i(x_i)$ on R, to show that the properties of the Fourier transform of the usual W-spaces also hold in this generalized W-spaces. In this note, notations based on [3].

Let $\rho_i(x_i)$ be strictly convex C^1-function on R such that $\rho_i(x_i)/|x_i|\to\infty$ as $|x_i|\to\infty$. The dual function $\rho_i^*(\eta_i)$ of $\rho_i(x_i)$ is defined by $\rho_i^*(\eta_i)=\operatorname*{Max}_{x_i}[\eta_i x_i-\rho_i(x_i)]$. We use the notations $\exp\rho(x)=\exp[\rho_1(x_1)+\cdots+\rho_n(x_n)]$ and $\exp\rho^*(\eta)=\exp[\rho_1^*(\eta_1)+\cdots+\rho_n^*(\eta_n)]$ for $\rho(x)=[\rho_1(x_1),\cdots,\rho_n(x_n)]$ and $\rho^*(\eta)=[\rho_1^*(\eta_1),\cdots\cdots,\rho_n^*(\eta_n)]$.

Since $\eta_i=\rho_i'(x_i)$ is strictly increasing, there exists the inverse function $x_i=\rho'^{-1}_i(\eta_i)$ which is a strictly increasing function. Let $\rho_i'(x_i^0)=0$ and $\rho_i'^{-1}(\eta_i^0)=0$.

LEMMA. *Defining* $\psi_i(\eta_i)=\int_{\eta_i^0}^{\eta_i}\rho_i'^{-1}(v_i)dv_i-\rho_i(0)$, *we have*

$$\rho_i(x_i)+\psi_i(\eta_i)\geq\eta_i x_i,$$

for any η_i *and* x_i.

Moreover for any η_i *and* x_i *such that* $\eta_i=\rho_i'(x_i)$, *we have*

$$\rho_i(x_i)+\psi_i(\eta_i)=\eta_i x_i.$$

Proof. From $\rho_i(x_i)=\int_{x_i^0}^{x_i}\rho_i'(u_i)du_i+\rho_i(x_i^0)$

and

$$\psi_i(\eta_i)=\int_{\eta_i^0}^{\eta_i}\rho_i'^{-1}(v_i)dv_i-\rho_i(0),$$

we have

$$\rho_i(x_i)+\psi_i(\eta_i)\geq\eta_i x_i+\rho_i(x_i^0)-\rho_i(0)+\int_{\eta_i^0}^{0}\rho_i'(u_i)du_i=\eta_i x_i.$$

The equality obviously holds in case $\eta_i=\rho_i'(x_i)$.

THEOREM 1. $\psi(\eta)=\rho^*(\eta)$.

Proof. For any given η_i, taking $x_i=\rho_i'^{-1}(\eta_i)$,

$$\psi_i(\eta_i)=\eta_i x_i-\rho_i(x_i)=\mathrm{Max}_{x_i}[\eta_i x_i-\rho_i(x_i)]=\rho_i^*(\eta_i).$$

THEOREM 2. *If $\rho(x)\geqslant\rho^0(x)$ for $x\geq x^1$ (or $x\leq x^1$), then $\rho^*(\eta)\leqslant\rho^{0*}(\eta)$ for $\eta\geq\eta^1$ (or $\eta\leqslant\eta^1$) where $\eta^1=(\rho_1'(x_1^1),\cdots,\rho_n'(x_n^1))$.*

Proof. Taking $\eta_i=\rho_i'(x_i)$, by the above Lemma,

$$\rho_i(x_i)+\rho_i^*(\eta_i)=\eta_i x_i\leq\rho_i^0(x_i)+\rho_i^{0*}(\eta_i).$$

From the above results, for any η, there exists only one x and for any x, there exists only one η such that $\rho(x)+\rho^*(\eta)=(\eta_i x_i)$. Hence we have the dual relation $\rho_i(x_i)=\underset{\eta_i}{\mathrm{Max}}[x_i\eta_i-\rho_i^*(\eta_i)]$ [**2**].

We now define the generalized W-spaces.

For any $a=(a_1,\cdots\cdots,a_n)>0$, we denote by $W_{\rho,a}$ the space of all C^∞-functions $\varphi(x)$ on R^n such that

$$|\partial^q\varphi(x)|\leqslant C_{qa'}'e^{-\rho(a'x)},\quad (0\leqslant|q|<\infty)\ \text{for any } a'<a,$$

where the C_{qa}' are constants depending on φ.

$W_{\rho,a}$ is a linear space with the topology in terms of the sequence of norms

$$\|\varphi\|_p=\sup_{|\alpha|\leqslant p}\sup_{x\in R^n} e^{\rho[a(1-\frac{1}{p})x]}|\partial^\alpha\varphi(x)|,\quad p=1,2,\cdots\cdots,$$

and we know that $W_{\rho,a}$ is a perfect space [**1**].

We denote by W_ρ the union space of the spaces $W_{\rho,a}$ $(0<a\in R^n)$.

From the Theorem 1, $\rho_i^*(\eta_i)$ is a strictly convex function on R and $\rho_i^{*(\eta_i)}/_{|\eta_i|}\to\infty$ as $|\eta_i|\to\infty$ [**2**].

For any $b=(b_1,\cdots,b_n)>0$, we denote by $W^{\rho*,b}$ the set of all functions extendable into entire functions $\varphi(z)$ on C^n such that

$$(1+|z|^k)|\varphi(z)|\leqslant C_{kb'}e^{\rho*(b'\eta)},\quad (0\leqslant k<\infty)\ \text{for any } b'>b$$

where the $C_{kb'}$ are constants depending on φ.

$W^{\rho*,b}$ is a linear space with the topology in terms of the sequence of norms

$$\|\varphi\|_p=\sup_{z\in C^n}(1+|z|)^p e^{-\rho*[b(1+\frac{1}{p})\eta]}|\varphi(z)|,\ p=1,2\cdots\cdots,$$

and a perfect space [**1**].

We denote by $W^{\rho*}$ the union space of the spaces $W^{\rho*,b}$ $(0<b\in R^n)$ and $W_{\rho,a}^{\rho*,b}$ the space of all the functions in $W^{\rho*,b}$ such that

$$|\varphi(x+i\eta)|\leqslant C_{a'b'}e^{-\rho(a'x)+\rho*(b'\eta)},\ \text{for any } a'<a,\ b'>b.$$

In $W_{\rho,a}^{\rho*,b}$, a topology is defined in terms of the norms

$$\|\varphi\|_p=\sup_{z\in C^n}|\varphi(z)|e^{\rho[a(1-\frac{1}{p})x]-\rho*[b(1+\frac{1}{p})\eta]},\quad p=1,2,\cdots\cdots.$$

We denote by $W_\rho^{\rho*}$ the union space of $W_{\rho,a}^{\rho*,b}$ $(0<a\in R^n,\ 0<b\in R^n)$.
If each $\rho_i(x_i)$ has the minimum 0 at O, our definitions of W-spaces are same as the definitions of the usual W-spaces.

We obtain the following results with the proofs similar to those in [1].

THEOREM 3. *For any* $a>0,\ b>0$,

$$\mathcal{F}[W_{\rho,a}]=W^{\rho*,1/a},\quad \mathcal{F}[W^{\rho*,b}]=W_{\rho,1/b}.$$

COROLLARY. $\mathcal{F}[W_\rho]=W^{\rho*}$, $\mathcal{F}[W^{\rho*}]=W_\rho$.

THEOREM 4. $\mathcal{F}[W_{b,\ a}^{\rho 0*,b}]=W_{\rho 0,\ 1/b}^{\rho*,\ 1/a}$.

COROLLARY. $\mathcal{F}W_\rho^{\rho*}=W_{\rho 0}^{\rho*}$.

If u is in ${}_\rho S$, the Fourier-Laplace transform of u is an entire function $F_u(\xi+i\eta)$ such that for any N and m, we can take a constant $C_{N,m}$, with which we have

$$(1+|\xi+i\eta|)^N|\partial_\zeta{}^m F_u(\xi+i\eta)|\leqslant C_{N,m}e^{\rho*(\eta)}\ [2].$$

Hence we obtain the following

THEOREM 5. *For u in ${}_\rho S$, the Fourier transform of u belongs $W^{\rho*,b}$, for any $b>1=(1,\cdots\cdots,1)$.*

References

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