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COMPACTIFICATIONS OF LOCALLY COMPACT σ-COMPACT SPACES

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1. Statement of results

In this note, all spaces are Hausdorff. Our starting point is

(A) *Let X be a countably compact space and let V be an open subset of X. If both V and $X-V$ are metacompact, then X is compact.*

If V happens to be dense in the countably compact space X, we shall call X a countable compactification of V by $X-V$. In this terminology, (A) may berephrased as

(A′) *Every countable compactification of a locally compact metacompact space by a metacompact set is a compactification.*

Since every σ-compact space is automatically metacompact, (A′) serves as a compactification theorem for locally compact σ-compact spaces. A consequence of this version of (A′) is the following

(B) *If every σ-compact open subset has metacompact boundary in a normal space X, then the intersection of all free maximal ideals of $C(X)$ is identical with $C_K(X)$.*

Here, $C(X)$ denotes the ring of real valued functions continuous on X, while $C_K(X)$ is the subring of $C(X)$ consisting of those functions having compact (possibly empty) supports. A maximal ideal of $C(X)$ is free if, for each point of X, it contains a function not vanishing at that point.

2. Proofs

By Arens and Dugundji [2, p. 171], a space is compact if it is countably compact and metacompact. Therefore, (A) will follow if we prove that X is metacompact. To do this, let $\mathcal{G}$ be an open cover of X and let $\mathcal{G}\cap V$ denote the open cover of V cosisting of those sets which are members of $\mathcal{G}$ intersected with V. By metacompactness of V, $\mathcal{G}\cap V$ has a refinement $\mathcal{H}$ which is a point finite cover of V by open subsets of V. Since V is open in

X, it follows that $\mathscr{H}$ is a point finite collection of open sets in X. On the other hand, $X-V$ is countably compact as it is closed in the countably compact space X. Thus the metacompact space $X-V$ is compact by the Arens-Dugundji theorem, implying that $\mathscr{G}$ has a finite subcollection $\mathscr{F}$ covering $X-V$. Metacompactness of X follows as it is now clear that $\mathscr{F}\cup\mathscr{H}$ is a point finite open cover of X refining $\mathscr{G}$. This completes the proof of (A).

To verify (A'), we only need observe that a locally compact space is open in any space containing it as dense subset.

Now to the proof of (B). It is shown in [3] that if a nonzero function f lies in all free maximal ideals of $C(X)$ then $Coz(f)$=the set of points at which f does not vanish is locally compact, σ-compact and totally bounded relative to every uniform structure admissible for X. Since this and the normality of X imply that $Coz(f)$ has countably compact closure, the support of f is a countable compactification of $Coz(f)$ by its boundary. But $Coz(f)$ has metacompact boundary by hypothesis and the support of f must be compact by (A'). This completes the proof of (B) as the other inclusion is trivial.

3. An example

Pseudocompactness cannot replace the countable compactness condition for X in (A). Indeed, if the continuum hypothesis is true, a pseudocompact space may fail to be compact even if it is the disjoint union of a countable discrete space and a hereditarily paracompact set. To see this, let P denote the set of all P-points of $\beta N-N$. That is, P is the set of those points p of βN-N such that every function in $C(\beta N-N)$ is stationary on some neighborhood of p, where βN denotes the Stone-Cech compactification of the integers N. It is known [1, 6V] that the continuum hypothesis implies that P is dense in $\beta N-N$. Thus, $Y=N\cup P$ is pseudocompact as it meets every zero set of $\beta Y=\beta N$. However, Y can not be compact as there are non P-points in βN-N.

To prove that P is hereditarily paracompact, let Q be an arbitrary subset of P and let $\mathscr{G}$ be an open cover of Q. Since Q is a subspace of βN, $\mathscr{G}$ has a refinement by open sets of the form $Q\cap Coz(f)$, $f\in C(\beta N)$, which we shall call basic open sets of Q. Thus, we may suppose with no loss of generality that members of $\mathscr{G}$ are basic open sets. Since $C(\beta N)$ has exactly continuously many functions, we may also suppose by using the continuum hypothesis that $\mathcal{Q}$ is well ordered so that each initial segment of $\mathcal{Q}$ is a countable collection. For each B in $\mathcal{Q}$, let B^* be the union of those members of $\mathcal{Q}$ which precede B and let $B'=B-B^*$. We propose that these B' form an open cover $\mathcal{Q}'$ of Q. First, each B^* is a basic open set of Q. This is true because cozero sets are closed under countable unions in any space. But then B^* is closed in Q as otherwise Q would contain a non P-point of βN-N. It follows that members of $\mathcal{Q}'$ are open in Q. Next, if x is a point of Q, let B be the first member of $\mathcal{Q}$ containing x. Then x is in $B-B^*=B'$, and $\mathcal{Q}'$ covers Q. Since $B'\subset B$, the open cover $\mathcal{Q}'$ refines $\mathcal{Q}$. But members of $\mathcal{Q}'$ are pairwise disjoint by construction and it follows that P is hereditarily paracompact.

The normality of X in the hypothesis of (B) is used to assure that the support of f is countably compact if f is in the intersection of free maximal ideals of $C(X)$. This, however, is superfluous in a sense. As in Corollary 2 of [3, Lemma 2], it suffices to assume that countable discrete closed sets are C-embedded in X instead of supposing that X be normal. The example Y above shows that this latter condition can not be further weakened to the condition that countable discrete closed sets be C^*-embedded in X. In Y, boundary of any subset of Y is contained in P, which is already seen to be hereditarily paracompact. Thus, Y satisfies the condition that σ-compact open sets have metacompact boundary. On the other hand, it is readily seen that an infinite discrete set can be closed in Y only if it is contained in P. Therefore, every countable discrete subset of Y is C^*-embedded in Y because countable subsets of $\beta N-N$ are C^*-embedded in βN. Neverthless, the function f defined by $f(n)=1/n$ for n in N and $f(x)=0$ for x in P has the noncompact set Y as support although f lies in all free maximal ideals of $C(X)$ by the Gelfand-Kolmogoroff theorem [1].

References

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