Bull. Korean Math. Soc.
Vol. 12, No. 2, 1975

HYPERSURFACES IN MANIFOLDS WITH SASAKIAN 3-STRUCTURE

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§0. Introduction

Yano and Okumura [6] have defined the concept of an (f, g, u, v, λ)-structure in an even dimensional Riemannian manifold. Hypersurfaces with this structure in a Sasakian manifolds have been studied by Yano and Okumura [6], Yamaguchi [4] and Watanabe [3] and many authors. In particular, they proved that, if the (f, g, u, v, λ)-structure induced on a hypersurface of a Sasakian manifold is normal, the hypersurface is totally umbilical.

In this paper we define what we call a 3-structure in a hypersurface of a Sasakian 3-structure manifold and study the normalities of this 3-structure.

§1. Sasakian 3-structure manifold

Let $\tilde{M}$ be an n-dimensional differentiable manifold covered by a system of coordinate neighborhood $\{U; y^\kappa\}$, where here and in the sequel, the indices κ, $\lambda, \mu, \nu, \cdots$ run over the range $\{1, 2, \cdots, n\}$. In this differentiable manifold $\tilde{M}$, a set (ϕ, ξ, ρ) of three tensor fields ϕ, ξ and ρ of type $(1,1)$, $(1,0)$ and $(0,1)$ respectively is called an *almost contact structure*, if it satisfies the following conditions:

$$\phi_\lambda{}^\kappa\xi^\lambda=0, \qquad \phi_\lambda{}^\kappa\rho_\kappa=0, \qquad \xi^\lambda\rho_\lambda=1, \tag{1.1}$$

$$\phi_\lambda{}^\kappa\phi_\nu{}^\lambda=-\delta_\nu^\kappa+\rho_\nu\xi^\kappa, \tag{1.2}$$

where $\phi_\lambda{}^\kappa$ is necessarily of rank $n-1$.

When a manifold admits an almost contact structure, it is called an *almost contact manifold* and is necessarily of odd-dimensional. There exists in any almost contact manifold a Riemannian metric $\tilde{g}_{\lambda\kappa}$ such that

$$\tilde{g}_{\lambda\kappa}\xi^\lambda=\rho_\kappa, \qquad \tilde{g}_{\lambda\kappa}\phi_\mu{}^\lambda\phi_\nu{}^\kappa=\tilde{g}_{\mu\nu}-\xi_\mu\xi_\nu, \tag{1.3}$$

and such a Riemannian metric $\tilde{g}_{\lambda\kappa}$ is called a *Riemannian metric associated* with the given almost contact structure (ϕ, ξ, ρ). An almost contact manifold is called an *almost contact Riemannian manifold*, when it is endowed with an associated Riemannian metric $\tilde{g}_{\lambda\kappa}$.

An almost contact Riemannian manifold is called a Sasakian manifold (or a normal contact Riemannian manifold) if a certain tensor field constructed from the structure $(\phi, \xi, \rho, \tilde{g})$ vanishes. However, an almost contact Riemannian manifold is normal if and only if the conditions

$$(1.4)\qquad \nabla_\lambda \xi_\kappa = \phi_{\lambda\kappa}, \quad \nabla_\mu \phi_{\lambda\kappa} = \xi_\lambda \tilde{g}_{\kappa\mu} - \xi_\kappa \tilde{g}_{\lambda\mu}, \quad \phi_{\lambda\kappa} = \tilde{g}_{\kappa\mu} \phi_\lambda{}^\mu$$

are satisfied, where in the following we use a notation ξ_λ in stead of ρ_λ. In a Riemannian manifold $(\tilde{M}, \tilde{g})$, a Sasakian structure $(\phi, \xi, \rho, \tilde{g})$ is sometimes denoted simply by ξ.

We now assume that there are three Sasakian structures $(\phi, \xi, \tilde{g})$, $(\psi, \eta, \tilde{g})$ and $(\theta, \zeta, \tilde{g})$ in $\tilde{M}$. Then, such a set $\{\xi, \eta, \zeta\}$ of three Sasakian structures ξ, η and ζ is called a Sasakian 3-structure (or normal contact metric 3-structure) if it satisfies the following conditions:

$$(1.5)\qquad \xi^\lambda \eta_\lambda = \eta^\lambda \zeta_\lambda = \zeta^\eta \xi_\lambda = 0,$$

$$(1.6)\qquad \psi_\lambda{}^\kappa \zeta^\lambda = -\theta_\lambda{}^\kappa \eta^\lambda = \xi^\kappa, \quad \theta_\lambda{}^\kappa \xi^\lambda = -\phi_\lambda{}^\kappa \zeta^\lambda = \eta^\kappa, \quad \phi_\lambda{}^\kappa \eta^\lambda = -\psi_\lambda{}^\kappa \xi^\lambda = \zeta^\kappa,$$

$$(1.7)\qquad \psi_\lambda{}^\mu \theta_\mu{}^\kappa = \phi_\lambda{}^\kappa + \eta_\lambda \zeta^\kappa, \quad \theta_\lambda{}^\mu \phi_\mu{}^\kappa = \psi_\lambda{}^\kappa + \zeta_\lambda \xi^\kappa, \quad \phi_\lambda{}^\mu \psi_\mu{}^\kappa = \theta_\lambda{}^\kappa + \xi_\lambda \eta^\kappa.$$

In such a case, the manifold $\tilde{M}$ is necessarily of dimension $n=4m+3$ $(m \geq 0)$ (cf. [2]) and is called a *Sasakian 3-structure manifold*.

§2. Surfaces in Sasakian 3-structure manifolds

In this section, we consider hypersurfaces in a Sasakian 3-structure manifold $\tilde{M}$. Let M be a $(4m+2)$-dimensional differentiable manifold covered by a system of coordinate neighborhood $\{U; x^h\}$, where here and in the sequel, the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, 4m+2\}$, and assume that M is differentiably immersed in M as a hypersurface by immersion $i: M \longrightarrow \tilde{M}$, which is represented by the equations

$$y^\kappa = y^\kappa (x^i)$$

in each coordinated neighborhood $\tilde{U}$ of $\tilde{M}$. If we put

$$B_i{}^\kappa = \partial_i y^\kappa \qquad (\partial_i = \partial / \partial x^i),$$

then $B_i{}^\kappa$ define a local vector field in $\tilde{U}$ for each fixed index i and $4m+2$ vector fields $B_i{}^\kappa$ span the tangent plane to M at each point of $\tilde{U}$. On putting

$$g_{ji}=g_{\lambda\kappa}B_j^{\lambda}B_i^{\kappa},$$

we see that g_{ji} define in M a Riemannian metric which is called the induced metric.

As is well known [1], a contact manifold is always orientable. We assume that the hypersurface M is also orientable and $4m+2$ tangent vectors B_i^{κ} are chosen in such a way that $B_1^{\lambda}, \cdots, B_{4m+2}^{\lambda}$ form a frame of positive orientation in M. Then we can choose a globally defined field of unit normal vectors C^{λ} in such a way that $4m+3$ vectors C^{λ}, $B_1^{\lambda}, \cdots, B_{4m+2}^{\lambda}$ form a frame of positive orien tation in $\tilde{M}$. Then, we get

$$\tilde{g}_{\lambda\kappa}B_i^{\lambda}C^{\kappa}=0, \qquad C^{\lambda}C_{\lambda}=1,$$
$$B^i{}_{\lambda}B_j^{\lambda}=\delta_j^i, \qquad B^i{}_{\lambda}B_i^{\kappa}=\delta_{\lambda}^{\kappa}-C_{\lambda}C^{\kappa},$$

where we have put

$$B^j{}_{\lambda}=\tilde{g}_{\lambda\kappa}B_i^{\kappa}g^{ji}, \qquad C_{\lambda}=\tilde{g}_{\lambda\kappa}C^{\kappa}.$$

The transforms $\phi_{\lambda}^{\kappa}B_i^{\lambda}$, $\psi_{\lambda}^{\kappa}B_i^{\lambda}$ and $\theta_{\lambda}^{\kappa}B_i^{\lambda}$ of B_i^{λ} can be expressed respectively as

(2.2)
$$\phi_{\lambda}^{\kappa}B_i^{\lambda}=\Phi_i^{h}B_h^{\kappa}+u_iC^{\kappa},$$
$$\psi_{\lambda}^{\kappa}B_i^{\lambda}=\Psi_i^{h}B_h^{\kappa}+v_iC^{\kappa},$$
$$\theta_{\lambda}^{\kappa}B_i^{\lambda}=\Theta_i^{h}B_h^{\kappa}+w_iC^{\kappa},$$

where Φ_i^h, Ψ_i^h and Θ_i^h are tensor fields of type $(1,1)$, and u_i, v_i and w_i 1-form of M.

The transforms of C^{λ} by ϕ_{λ}^{κ}, ψ_{λ}^{κ} and $\theta_{\lambda}^{\kappa}$ can be put respectively

(2.3)
$$\phi_{\lambda}^{\kappa}C^{\lambda}=-u^iB_i^{\kappa}, \qquad \psi_{\lambda}^{\kappa}C^{\lambda}=-v^iB_i^{\kappa}, \qquad \theta_{\lambda}^{\kappa}C^{\lambda}=-w^iB_i^{\kappa},$$

where $u^i=g^{ji}u_j$, $v^i=g^{ji}v_j$ and $w^i=g^{ji}w_j$.

Taking account of (2.2) and (2.3), we have

(2.4)
$$\Phi_j^{i}=B^i{}_{\lambda}\phi_{\mu}^{\lambda}B_j^{\mu}, \qquad \Psi_j^{i}=B^i{}_{\lambda}\psi_{\mu}^{\lambda}B_j^{\mu}, \qquad \Theta_j^{i}=B^i{}_{\lambda}\theta_{\mu}^{\lambda}B_j^{\mu},$$

(2.5)
$$u_j=B_j^{\lambda}\phi_{\lambda}^{\mu}C_{\mu}, \qquad v_j=B_j^{\lambda}\psi_{\lambda}^{\mu}C_{\mu}, \qquad w_j=B_j^{\lambda}\theta_{\lambda}^{\mu}C_{\mu}.$$

If we put

(2.6)
$$\xi^{\kappa}=B_i^{\kappa}\xi^i+\alpha C^{\kappa}, \quad \eta^{\kappa}=B_i^{\kappa}\eta^i+\beta C^{\kappa}, \quad \zeta^{\kappa}=B_i^{\kappa}\zeta^i+\gamma C^{\kappa},$$

then by virtue of (1.1), (1.2), (2.4), (2.5) and (2.6) we easily find the following equations (2.7)-(2.10):

(2.7) $$\Phi_{ji}=\Phi_j{}^t g_{ti}=-\Phi_{ij},$$

(2.8) $$\Phi_j{}^h\Phi_h{}^i=-\delta_j^i+u_ju^i+\xi_j\xi^i,$$

(2.9) $$\xi_i\Phi_j{}^i=-\alpha u_j, \quad u_i\Phi_j{}^i=\alpha\xi_j,$$

(2.10) $$u^iu_i=\xi^i\xi_i=1-\alpha^2, \quad u^i\xi_i=0,$$

and for another two Sasakian structures the similar relations are obtained.

The equations (2.8)-(2.10) show that $(\Phi, g, u, \xi, \alpha)$ is a so called (f, g, u, v, λ)-structure in M (See [6]). Thus we have now three (f, g, u, v, λ)-structures $(\Phi, g, u, \xi, \alpha)$, $(\Psi, g, v, \eta, \beta)$ and $(\Theta, g, w, \zeta, \gamma)$ in M.

Applying again Φ, Ψ and Θ to (2.2) and taking account of (1.7), (2.3) and (2.6), we get

(2.11)
$$\begin{aligned}
&\Psi_j{}^h\Theta_h{}^i=+\Phi_j{}^i+v_jw^i+\eta_j\zeta^i, && \Theta_j{}^h\Psi_h{}^i=-\Phi_j{}^i+w_jv^i+\zeta_j\eta^i,\\
&\Theta_j{}^h\Phi_h{}^i=+\Psi_j{}^i+w_ju^i+\zeta_j\xi^i, && \Phi_j{}^h\Theta_h{}^i=-\Psi_j{}^i+u_jw^i+\xi_j\zeta^i,\\
&\Phi_j{}^h\Psi_h{}^i=+\Theta_j{}^i+u_jv^i+\xi_j\eta^i, && \Psi_j{}^h\Phi_h{}^i=-\Theta_j{}^i+v_ju^i+\eta_j\xi^i,
\end{aligned}$$

(2.12)
$$\begin{aligned}
&v_i\Theta_j{}^i=-u_j+\beta\zeta_j, && w_i\Psi_j{}^i=u_j+\gamma\eta_j,\\
&w_i\Phi_j{}^i=-v_j+\gamma\xi_j, && u_i\Theta_j{}^i=v_j+\alpha\zeta_j,\\
&u_i\Psi_j{}^i=-w_j+\alpha\eta_j, && v_i\Phi_j{}^i=w_j+\beta\xi_j.
\end{aligned}$$

Applying again ϕ, ψ and θ to (2.3) and taking account of (1.7), (2.2), (2.3) and (2.6), we find

(2.13) $$u^iv_i=-\alpha\beta, \quad v^iw_i=-\beta\gamma, \quad w^iu_i=-\gamma\alpha.$$

Applying ϕ, ψ and θ to (2.6) and using (1.6), (2.2), (2.3) and (2.6), we obtain

(2.14)
$$\begin{aligned}
&\eta_i\Theta_j{}^i=\xi_j-\beta w_j, && \zeta_i\Psi_j{}^i=-\xi_j-\gamma v_j,\\
&\zeta_i\Phi_j{}^i=\eta_j-\gamma u_j, && \xi_i\Theta_j{}^i=-\eta_j-\alpha w_j,\\
&\xi_i\Psi_j{}^i=\zeta_j-\alpha v_j, && \eta_i\Phi_j{}^i=-\zeta_j-\beta u_j,
\end{aligned}$$

(2.15) $$v^i\zeta_i=-w^i\eta_i=\alpha, \quad w^i\xi_i=-u^i\zeta_i=\beta, \quad u^i\eta_i=-v^i\xi_i=\gamma.$$

The triple $\{(\Phi, g, u, \xi, \alpha), (\Psi, g, v, \eta, \beta), (\Theta, g, w, \zeta, \gamma)\}$ of (f, g, u, v, λ)-structures satisfying (2.11)-(2.15) is called a *3-structure*. We denote by $\{{}_\mu{}^\lambda{}_\nu\}$ the Christoffel symbols constructed from the given Riemannian metric $\tilde{g}_{\lambda\kappa}$ in $\tilde{M}$ and by $\{{}_j{}^h{}_i\}$ those constructed from the metric g_{ji} induced in the hypersurface M. We denote by h_{ji} the second fundamental tensor of the hypersurface M and put $h^i{}_j = g^{ik}h_{kj}$. Then, the equations of Gauss and Weingarten are given respectively by

$$\nabla_j B_i{}^\lambda = \partial_j B_i{}^\lambda + \{{}_\mu{}^\lambda{}_\nu\} B_j{}^\nu B_i{}^\mu - \{{}_j{}^h{}_i\} B_h{}^\lambda = h_{ji} C^\lambda, \tag{2.16}$$

$$\nabla_j C^\lambda = \partial_j C^\lambda + \{{}_\mu{}^\lambda{}_\nu\} B_j{}^\nu C^\mu = -h_j{}^i B_i{}^\lambda. \tag{2.17}$$

Differentiating (2.4), (2.5) and (2.6) covariantly along M and taking account of (2.16) and (2.17), we have

$$\nabla_j \Phi_i{}^h = -h_{ji}u^h + h_j{}^h u_i - g_{ji}\xi^h + \delta_j{}^h \xi_i, \tag{2.18}$$

$$\nabla_j u_i = -h_{ji}\Phi_i{}^t - \alpha g_{ji}, \tag{2.19}$$

$$\nabla_j \xi_i = \Phi_{ji} + \alpha h_{ji}, \tag{2.20}$$

and for another two Sasakian structures the similar relations are obtained.

§3. Hypersurfaces with 3-structure

As preliminalies, we recall the definitions of quasinormal and normal of an (f, g, u, v, λ)-structure.

We now put

$$\begin{aligned} S[\Phi, \Phi]_{ji}{}^h &= [\Phi, \Phi]_{ji}{}^h + (\nabla_j u_i - \nabla_i u_j)u^h + (\nabla_j \xi_i - \nabla_i \xi_j)\xi^h, \\ S[\Psi, \Theta]_{ji}{}^h &= [\Psi, \Theta]_{ji}{}^h + [\nabla_j v_i - \nabla_i v_j)w^h + (\nabla_i w_j - \nabla_i w_j)v^h \\ &\quad + (\nabla_j \eta_i - \nabla_i \eta_j)\zeta^h + (\nabla_j \zeta_i - \nabla_i \zeta_j)\eta^h, \end{aligned} \tag{3.1}$$

where $[\Phi, \Phi]$ is the Nijenhuis tensor formed with Φ and $[\Psi, \Theta]$ the Nijenhuis tensor formed with Ψ, Θ respectively. Similarily, we define $S[\Psi, \Psi]$, $S[\Theta, \Theta]$, $S[\Theta, \Phi]$ and $S[\Phi, \Psi]$ for the other tensors.

An (f, g, u, v, λ)-structure $(\Phi, g, u, \xi, \alpha)$ is said to be *quasi-normal* if the condition.

$$S[\Phi, \Phi]_{jih} - (\Phi_j{}^t \Phi_{tih} - \Phi_i{}^t \Phi_{tjh}) = 0 \tag{3.2}$$

is satisfied, where

$$\Phi_{jih} = \nabla_j \Phi_{ih} - \nabla_i \Phi_{hj} - \nabla_h \Phi_{ji}. \tag{3.3}$$

The structure $(\Phi, g, u, \xi, \alpha)$ is said to be *normal* if this structure satisfies

$$S[\Phi, \Phi] = 0. \tag{3.4}$$

In the following we study some properties on a hypersurface with the induced 3-structure of a manifold with Sasakian 3-structure.

Substituting (2.18), (2.19) and (2.20) into (3.1), we get

(3.5) $$S[\Phi, \Phi]_{ji}{}^h = (\Phi_j{}^t h_t{}^h - h_j{}^t \Phi_t{}^h) u_i - (\Phi_i{}^t h_t{}^h - h_i{}^t \Phi_t{}^h) u_j,$$

(3.6) $$S[\Psi, \Theta]_{ji}{}^h = (\Psi_j{}^t h_t{}^h - h_j{}^t \Psi_t{}^h) w_i - (\Psi_i{}^t h_t{}^h - h_i{}^t \Psi_t{}^h) w_j + (\Theta_j{}^t h_t{}^h - h_j{}^t \Theta_t{}^h) v_i - (\Theta_i{}^t h_t{}^h - h_i{}^t \Theta_t{}^h) v_j.$$

By the first equation of (3.5), (3.4) is equivalent to the commutativity of Φ and h on a hypersurface of a Sasakian manifold.

The following Lemma A is known ([6]).

LEMMA A. *Let $M(>2)$ be an orientable connected hypersurface of a Sasakian manifold $\tilde{M}$. If one of Φ, Ψ and Θ commute h and $\alpha^2 \neq 1$ (resp. $\beta^2 \neq 1$ or $\gamma^2 \neq 1$) almost everywhere, then the hypersurface M is totally umbilical.*

Substituting (2.18) into (3.4), we find $\Phi_{jih}=0$. Thus the equation (3.3) shows that the structure $(\Phi, g, u, \xi, \alpha)$ is normal on the hypersurface M.

So we have the following from Lemma A and (3.5)

PROPOSITION 3.1. *Let M be a hypersurface with a 3-structure $\{(\Phi, g, u, \xi, \alpha), (\Psi, g, v, \eta, \beta), (\Theta, g, w, \zeta, \gamma)\}$ of a Sasakian manifoled. If one of $3(f, g, u, v, \lambda)$-structures $(\Phi, g, u, \xi, \alpha)$, $(\Psi, g, v, \eta, \beta)$ and $(\Theta, g, w, \zeta, \gamma)$ is a normal on M, then the others are so also.*

PROPOSITION 3.2. *Under the same assumptions as those in Lemma A, all of $S[\Phi, \Phi]$, $S[\Psi, \Psi]$, $S[\Theta, \Theta]$, $S[\Psi, \Theta]$, $S[\Theta, \Phi]$ and $S[\Phi, \Psi]$ are vanished.*

Now we prove

PROPOSITION 3.3. *If the vectors u^h, v^h and w^h for the induced 3-structure on a hypersurface of a Sasakian 3-structure manifold are linearly independent alm ost everywhere, and if $S[\Psi, \Theta]=0$, then Ψ and Θ are normal.*

Proof. From the second equation of (3.2), we have

(3.7) $$(\Psi_j{}^t h_t{}^h - h_j{}^t \Psi_t{}^h) w_i + (\Theta_j{}^t h_t{}^h - h_j{}^t \Theta_t{}^h) v_i = (\Psi_i{}^t h_t{}^h - h_i{}^t \Psi_t{}^h) w_j + (\Theta_i{}^t h_t{}^h - h_i{}^t \Theta_t{}^h) v_j.$$

Transvecting (3.7) with v^i and w^i respectively and using (2.10) and (2.13), we obtain

(3.8) $$(\Psi_j{}^t h_{th} - h_j{}^t \Psi_{th})(-\beta\gamma) + (\Theta_j{}^t h_{th} - h_j{}^t \Theta_{th})(1-\beta^2) = \beta' v_j v_k + \gamma' w_j w_h,$$

(3.9) $$(\Psi_j{}^t h_{th} - h_j{}^t \Psi_{th})(1-\gamma^2) + (\Theta_j{}^t h_{th} - h_j{}^t \Theta_{th})(-\beta\gamma) = \beta'' v_j v_h + \gamma'' w_j w_h,$$

where β', β'', γ' and γ'' are defined respectively by

$$\beta' v_h = v^i(\Theta_i{}^t h_{th} - h_i{}^t \Theta_{th}), \quad \gamma' w_h = v^i(\Psi_i{}^t h_{th} - h_i{}^t \Psi_{th}),$$
$$\beta'' v_h = w^i(\Theta_i{}^t h_{th} - h_i{}^t \Theta_{th}), \quad \gamma'' w_h = w^i(\Psi_i{}^t h_{th} - h_i{}^t \Psi_{th}).$$

Eliminating the terms of $w_j w_h$ from (3.8) and (3.9), we get

$$[(1-\gamma^2)\gamma' + \beta\gamma\gamma''](\Psi_j{}^t h_{th} + \Psi_h{}^t h_{tj}) - [(1-\beta^2)\gamma'' + \beta\gamma\gamma']$$
$$\times (\Theta_j{}^t h_{th} + \Theta_h{}^t h_{tj}) = (\beta''\gamma' - \beta'\gamma'') v_j v_h.$$

from which, by transvecting g^{jh}, $(\beta''\gamma' - \beta'\gamma'')(1-\beta^2) = 0$.

Since v^h and w^h are linearly independent almost everywhere, i.e.,

$$\begin{vmatrix} 1-\beta^2 & -\beta\gamma \\ -\beta\gamma & 1-\gamma^2 \end{vmatrix} \neq 0 \quad \text{almost everywhere.}$$

This together with (3.8) and (3.9) show that Ψ and Θ commute with h. Hence Ψ and Θ are normal structure.

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