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SOME PROPERTIES OF $K\{M_p\}$ AND $Z\{M_p\}$ SPACES

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1. Introduction.

In this note, we obtain that translate is an isomorphism on the $K\{M_p\}$ and $Z\{M_p\}$ spaces under some conditions on $\{M_p\}$. A part of the results on $K\{M_p\}$ spaces is the same one of [4] without some conditions. We also deal some properties of the generalized W-spaces [3] and the relations with the usual W-spaces [1].

Throughout this note we use the terminology and notation in [1, 2, 3].

2. Translate on $K\{M_p\}$ and $Z\{M_p\}$ spaces.

If φ is in $K\{M_p\}$ or $Z\{M_p\}$ spaces and $h\in R^n$, the translate of φ by h is denoted by $\tau_h\varphi$.

We will only consider $K\{M_p\}$ and $Z\{M_p\}$ spaces that satisfy the following condition F.

F: *For each p, there is a $p'>p$ and $C_{p',h}>0$ such that $\tau_h M_p \geq C_{p',h} M_{p'}$ for all $h\in R^n$.*

The examples of the spaces satisfying the condition F are the space $\mathcal{S}$ of rapidly decreasing functions, $W_{M,a}$, $W^{\Omega,b}$ and $W^{\Omega,b}_{M,a}$ [1].

Also the spaces $W_{\rho,a}$, $W^{\rho*,b}$ and $W^{\rho 0*,b}_{\rho,a}$ introduced in [3], satisfy the condition F.

We know that $\tau_h : \mathcal{D}\to\mathcal{D}$ is an isomorphism [2].

Furthermore we have the following

THEOREM 1. *Let $\{M_p\}$ satisfy the condition F. Then the map $\tau_h : K\{M_p\}\to K\{M_p\}$ (or $Z\{M_p\}\to Z(M_p\})$ is an isomorphism, for each $h\in R^n$.*

Proof. Let us show that τ_h is continuous on $K\{M_p\}$. For $\varphi\in K\{M_p\}$,

$$\sup_{|\alpha|\leq p}\sup_x M_p(x)|D^\alpha\tau_h\varphi(x)| = \sup_{|\alpha|\leq p}\sup_x M_p(x-h)|D^\alpha\varphi(x)|$$
$$\leq C_{p',h}\sup_{|\alpha|\leq p'}\sup_x M_{p'}(x)|D^\alpha\varphi(x)|.$$

Hence $\|\tau_h\varphi\|_p\leq C_{p',h}\|\varphi\|_{p'}$, from which we conclude that τ_h is continuous.

Since $\tau_h^{-1}=\tau_{-h}$, τ_h^{-1} is continuous. The bijectivity of τ_h is clear. This completes the proof for $K\{M_p\}$.

For the space $Z\{M_p\}$, we can show that τ_h is an isomorphism by using the same method of the above argument.

COROLLARY 2. *Let $\{M_p\}$ satisfy the condition F and B be a bounded subset of $K\{Mp\}$ (or $Z\{M_p\}$). Then for any $\varepsilon>0$, the set $\{|\tau_h\varphi||h|\leq\varepsilon, \varphi\in B\}$ is also bounded in $K\{M_p\}$ (or $Z\{M_p\}$).*

From the above result, it follows that translate is an isomorphism on $W_{\rho,a}$, $W^{\rho*,b}$ and $W_{\rho,a}^{\rho 0*,b}$.

3. Some properties on the generalized W-spaces and the relations with the usual W-spaces.

DEFINITION. $\{M_p\}$ and $\{M_p'\}$ are *equivalent* iff there exist C_p and C'_p such that $0<C_p\leqq M_p/M_p' \leqq C'_p<\infty$.

If $\{M_p\}$ and $\{M'_p\}$ are equivalent, then $K\{M_p\}$ and $K\{M'_p\}$ ($Z\{M_p\}$ and $Z\{M'_p\}$) are the same linear topological spaces. If there is some x^1 such that $M_p=C_pM'_p$ for $|x|\geq|x^1|$ (or $|z|\geq|x^1|$), then $\{M_p\}$ and $\{M'_p\}$ are equivalent.

THEOREM 3. *If there is some x^1 such that $\rho(x)-\rho^1(x)$ is a constant for $|x|\geq|x^1|$, then $W_{\rho,a}$ and $W_{\rho 1,a}$ ($W^{\rho*,b}$ and $W^{\rho 1*,b}$) are the identical spaces. Furthermore if there is some x^0 such that $\rho^0(x)-\rho^2(x)$ is a constant for $|x|\geq|x^0|$, then $W_{\rho,a}^{\rho 0*,b}$ and $W_{\rho 1,a}^{\rho 2*,b}$ are the identical spaces.*

Proof. Taking $\eta_i=\rho_i'(x_i)=\rho^1{}_i'(x_i)$, we have

$$\rho_i(x_i)+\rho_i^*(\eta_i)=\eta_ix_i=\rho^1{}_i(x_i)+\rho^{1*}{}_i(\eta_i)$$

Hence the proof is straightforward.

REMARK. If $\rho(x)$ is symmetric and $\rho(0)=0$, then $W_{\rho 1,a}$ and $W^{\rho 1*,b}$ are same of our usual W-spaces. In particular, if $\rho^0(x)$ is symmetric and $\rho^0(0)=0$, then $W_{\rho 1,a}^{\rho 2*,b}$ are same of our usual W-spaces.

W_ρ, $W^{\rho*}$ and $W^{\rho o*}$ are evidently also the countable union space. Note that we have not defined a topology on the countable union space. But we can define the concepts of continuity and isomorphism on the countable union space [1].

THEOREM 4. *If there is some* x^1 *such that* $\rho(rx)\leq\rho^1(r'x)$ *for* $|x|\geq|x^1|$, *then the topology of* $W_{\rho 1}$ $(W^{\rho 1*})$ *is stronger than the topology of* W_ρ $(W^{\rho*})$ *in the sence for countable union space* [1].

Proof. By using [3, Th. 2], the proof is immediate.

DEFINITION. ρ and ρ^1 are ***equivalent*** iff there is some x^1 such that $\rho(rx)\leq\rho^1(r'x)\leq\rho(r''x)$ for $|x|\geq|x^1|$.

COROLLARY 5. *If* ρ *and* ρ^1 *are equivalent, then* W_{p^1} *and* W_ρ $(W^{\rho 1*}$ *and* $W^{\rho*})$ *are identical space.*

REMARK. In particular, if $\rho(x)$ is symmetric and $\rho(0)=0$, then $W_{\rho 1}$ and $W^{\rho 1*}$ are the same of our usual W-spaces [1].

In [3], our results are not true in general except that $\mathcal{F}[W^{\rho*,b}]=W_{\rho,1/b}$ and $\mathcal{F}[W^{\rho*}]=W_\rho$. Other Fourier images of the generalized W-spaces are incl-uded in the range spaces. But if there is some x^1 such that ρ and ρ^0 are sym metric for $|x|\geq|x^1|$, then our results are true.

References

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