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A NOTE ON THE ESSENTIAL NILPOTENCY

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An ideal L of a ring R is called *essentially nilpotent* if it contains a nilpotent ideal N of R which is essential in L, i.e., N has non-zero intersection with each non-zero ideal of R which is contained in L [2]. A nil right (left) ideal of a ring is called essentially right (left) nilpotent if it contains a nilpotent right (left) ideal which is essential in it. If N is an ideal of a ring R, N is called *left T-nilpotent* ("T" for transfinite) if, given any sequence $\{a_i\}$ of elements in N, there exists an n such that $a_1a_2\cdots a_n=0$. (*Right T-nilpotency* requires instead that $a_na_{n-1}\cdots a_1=0$.) [1].

Shock proved that a nil right ideal is essentially nilpotent if and only if it contains an essential right ideal which is left T-nilpotent [3]. In this paper, we show that an ideal L is essentially nilpotent if and only if L contains a left T-nilpotent ideal which is essential in L.

REMARK 1. H. Bass' exemple (5), p. 476 of [1] shows the existance of a left T-nilpotent ideal but not right T-nilpotent. Therefore if an ideal N is left T-nilpotent, then N is not nilpotent.

REMARK 2. Using the Sasiada's example (Let R denoted the ring generated over the integers by $x_1, x_2, \cdots$ with the relation $x_ix_j=0$ for $i\geqq j$.), J. Fisher showed that essential nilpotency does not imply left T-nilpotency [2].

LEMMA. *If an ideal L of R is left T-nilpotent, then L is essentially nilpotent.*

Proof. In [2].

THEOREM. *An ideal L of a ring is essentially nilpotent if and only if L contains a left T-nilpotent ideal which is essential in L.*

Proof. Let J be a nilpotent ideal of R which is essential in essentially nilpotent ideal L. Hence J is a left T-nilpotent ideal which is essential in L.

Conversely, let J be a left T-nilpotent ideal contained in L which is essential in L. By the previous lemma, the left T-nilpotent ideal J is essentially nilpotent. Hence J contains a nilpotent ideal N which is essential in J. N is also essential in L. Hence L is essentially nilpotent.

References

[1] H. Bass, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc., **95** (1960). 466–488.

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[3] R. C. Shock, *Essentially nilpotent rings*, Israel J. Math., **9** (1971), 180–185.

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