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ON INJECTIVE RINGS

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1. Introduction

In [1] E. P. Armendariz and S. A. Steinberg concerned regular self-injective rings with a polynomial identity. Since the maximal quotient ring of a ring with zero singular ideal is von Neumann regular and self-injective, this paper investigates the structure of a biregular self-injective ring R by looking at its prime ideals, essential ideals or closed ideals. The reader is referred to [2] for the definitions and basic properties.

Throughout this paper we assume that R is a biregular self-injective ring with unit and C is the center of R.

2. Properties of R

PROPOSITION 1. *Let A be an ideal of R. Then A is essential in R if and only if* $\mathrm{Ann}_l(A)=(0)$.

Proof. If A is essential in R and $x\in\mathrm{Ann}_l(A)$, then $xA=(0)$. Assume $xR\neq(0)$. Then $xR\cap A\neq(0)$ and there is a nonzero element a in A such that $a=xr$ for some $r\in R$. Since R is semprime and $aRa=xrRa\subseteq xA=(0)$, we have $a=0$. This contradicts the fact that $a\neq 0$. Therefore $xR=0$ and this yields that $\mathrm{Ann}_l\ (A)=(0)$. Conversely, if I is a nonzero right ideal of R, then $IA\neq(0)$. Since $IA\subset I\cap A$, A is essential in R.

PROPOSITION 2. *If I is a nonzero right ideal of R, then $I\cap C\neq(0)$*

Proof. Let x be a nonzero element of I. Then $xR=eR$ for some central idempotent e since R is biregular. Hence $e\in I\cap C$.

PROPOSITION 3. *Let I be an essential right ideal of R. Then I contains an essential two-sided ideal of R.*

Proof, Let U be the largest two-sided ideal of R which is contained in I.

Suppose U is not an essential ideal of R. Then $V=\mathrm{Ann}_l(U)\neq(0)$ and $(UV)^2=U(UV)V=(0)$. Since R is semiprime, $UV=(0)$. Let $W=VR$. By proposition 2, $I\cap W\cap C$ contains a nonzero element w. Then $w\notin U$ and $U\subset U+wR\subseteq I$.

PROPOSITION 4. *Let A be an ideal of R and J an ideal of C. Then* (a) $(A\cap C)R=A$ *and* (b) $JR\cap C=J$.

Proof. For any $a\in A$, $aR=eR$ for some central idempotent e. Since $a=er$ for some $r\in R$, $a=er\in(A\cap C)R$ and this proves (a). Let $c\in JR\cap C$. Then $c=\sum j_k r_k$, $j_k\in J$, $r_k\in R$. Since C is regular, $\sum j_k C=eC$ for some idempotent e contained in J. Thus $c=ec\in J$ and (b) is proved.

PROPOSITION 5. *Let P be an ideal of R and D an ideal of C. Then* (a) *P is prime in R if and only if $P\cap C$ is prime in C;* (b) *D is prime in C if and only if DR is prime in R.*

Proof. (a) Suppose P is prime in R and E and F are ideals of C such that $EF\subseteq P\cap C$. Then $(ER)(FR)=(EF)R\subseteq(P\cap C)R=P$. Therefore $ER\subseteq P$ or $FR\subseteq P$ and this implies $E\subseteq P\cap C$ or $F\subseteq P\cap C$. Conversely, let A and B be ideals of R such that $AB\subseteq P$. Since $P\cap C$ is prime in C and since $(A\cap C)(B\cap C)\subseteq AB\cap C\subseteq P\cap C$, $A\cap C\subseteq P\cap C$ or $B\cap C\subseteq P\cap C$. Hence $A\subseteq P$ or $B\subseteq P$. Similarly (b) can be proved.

PROPOSITION 6. *Let A be an ideal of R and J an ideal of C. Then* (a) *A is closed in R if and only if $A\cap C$ is closed in C;* (b) *J is closed in C if and only if JR is closed in R.*

Proof. Suppose $A\cap C$ is closed in C and B is an essential extension of A. For any $t\in B\cap C$ such that $t(B\cap C)\neq(0)$, there exists $a\in A$ such that $a=tb$ or some $b\in B$ since $tB\cap A\neq(0)$. Since $bR=eR$ for some central idepotent e of R, $e=bs$ for some $s\in R$ and $a=eu$ for some $u\in R$. Then $0\neq as=te\in t(B\cap C)\cap(A\cap C)$ and this means that $B\cap C$ is an essential extension of $A\cap C$. Since $A\cap C$. is closed in C, $A\cap C=B\cap C$ and $A=B$. This proves that A is closed in R. Conversely let D be an essential extension of $A\cap C$. For any $t\in DR$ such that $tDR\neq(0)$, $tDR=\mathrm{d}DR$ for some $d\in D$ and $tDR\cap A\supset dD\cap(A\cap C)\neq(0)$. Then DR is an essential extension of A and this implies $DR=A$ since A is closed in R. Hence $D=A\cap C$ and $A\cap C$ is closed in C. Similarly we can prove (b).

COROLLARY (a) *A is essential in R if and ondly if $A\cap C$ is essential in C;* (b) *J is essential in C if and only if JR is essential in R.*

We consider the set $L(R)$ of closed ideals of R and the set $L(C)$ of closed ideals of C. From proposition 4 and proposition 6, there is a 1-1 correspondence between $L(R)$ and $L(C)$ given by $A\rightarrow A\cap C$, $J\rightarrow JR$ where $A\in L(R)$ and $J\in L(C)$. Similarly there is a 1-1 correspondence between the essential ideals of R and the essential ideals of C, and there is a 1-1 correspondence between prime ideals of R and prime ideals of C. With an eye to the commutative theory, a biregular self-injective ring with unit can be characterized by the properties of closed, prime or essential ideals. Moreover a prime ideal in R is either essential or closed [3].

References

[1] E. P. Armendariz and S. A. Steinberg, *Regular self-injective rings with a polynomial identity,* Trans. Amer. Math. Soc. **190** (1974), 158-173.

[2] C. Faith, *Lectures on injective modules and quotient rings,* Springer-Verlag Lecture Notes, No. 49.

[3] K. R. Goodearl, *Prime ideals in regular self-injective rings,* Can. J. Math. Vol. XXV, No. 4, 1973, 829-839.

[4] N. Jacobson, *Structure of rings,* Amer. Math. Soc. Colloq. Publ., Vol. **37**, Amer. Math. Soc., Providence, R. I., 1964.

[5] J. Lambek, *Lectures on rings and modules,* Blaisdell, Waltham, Mass., 1966.

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