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ON QUOTIENT Π-IMAGES OF CERTAIN SEMIMETRIC SPACES

BY YOUNGSO KO

One of the fundamental directions in general topology is the determination of connections induced between classes of spaces by means of mappings of different types. Throughout this paper mappings are considered to be continuous and spaces to be T_1.

Let X be a topological space and d be a nonnegative real-valued function defined on $X\times X$ such that $d(x,y)=0$ if and only if $x=y$. Such a function d is called an *o-metric* [6] for X provided that a subset U of X is open if and only if $d(x, X-U)>0$ for each $x\in U$. An o-metric d is called a *symmetric* if $d(x,y)=d(y,x)$ for each x and y; Furthermore, a symmetric d is called a semimetric provided $x\in \mathrm{cl}(M)$ if and only if $d(x,M)=0$, for any $M\subset X$.

A mapping f of a space X o-metrized by an o-metric d onto a topological space Y is called a π-mapping [7] if for any open set U in Y and for any $y\in U$, there is a positive real number ϵ such that $S_d(f^{-1}y,\varepsilon)=\{x: d(f^{-1}y,x)<\varepsilon\}\subset f^{-1}U$.

A mapping $f: X\to Y$ is said to be *pseudo-open* [3] if for each $y\in Y$ and each neighborhood U of $f^{-1}y$, the $\mathrm{Int}(f(U))$ contains y.

Alexander [1] showed that a T_1-space Y is semimetrizable if and only if Y is a pseudo-open π-image of a metric space. considering the pseudo-open π-images of semimetric spaces, we have the following

THEOREM 1. *The pseudo-open π-image of a semimetric space is semimetriza-ale.*

Proof. Let (X, d) be a semimetric space and $f: X \to Y$ be a pseudo-open π-mapping. Let ρ be the quotient-distance defined by $\rho(x, y) = d(f^{-1}x, f^{-1}y)$, for each pair of points x, y in Y. Noting that f is a π-mapping, we can deduce that $\rho(x, y) = 0$ if and only if $x = y$.

Now it is sufficient to show that $x \in \mathrm{cl}(M)$ if and only if $\rho(x, M) = 0$. Suppose $\rho(x, M) = \varepsilon > 0$. This implies $d(f^{-1}x, f^{-1}M) = \varepsilon$, and hence $S_d(f^{-1}x, \varepsilon) \cap f^{-1}M = \phi$. Since d is a semimetric for X, $\mathrm{Int}[S_d(f^{-1}x, \varepsilon)] \supset f^{-1}x$. Hence $f(\mathrm{Int}[S_d(f^{-1}x, \varepsilon)])$ is a neighborhood of x, because f is pseudo-open, which is disjoint from M. Therefore $x \notin \mathrm{cl}(M)$.

Conversely, if $x \notin \mathrm{cl}(M)$, there exists an open set U containing x disjoint from M. Since f is a π-mapping, $d(f^{-1}x, f^{-1}M) > 0$.

It is known that a T_1-space X is developable if and only if X is semimetrizable by a semimetric under which all convergent sequences are Cauchy (Burke [4]). In the following we consider a developable space as a semimetric space by a semimetric under which all convergent sequences are Cauchy.

A mapping $f: X \to Y$ is said to be ***almost-open*** [9] if for each y in Y there exists a point $x(y)$ in $f^{-1}y$ with a local base $\mathscr{B}$ of $x(y)$ such that $f(B)$ is open in Y for each $B \in \mathscr{B}$.

Ponomarev [8] showed that a T_1-space Y is developable if and only if Y is an open π-image of a metric space.

The following theorem and its corollary are strengthening of this proposition. In the above proposition, the condition that Y be an open π-image of a metric space can be weakened in two senses; one is replacing metric space by developable space, and the other is replacing open π-mapping by almost-open π-mapping.

THEOREM 2. *The almost-open π-image of a developable space is developable.*

Proof. Let (X, d) be a semimetric space in which all convergent sequences ane Cauchy under the distance function d. Let $f: X \to Y$ be an almost-open π-mapping and ρ the quotient-distance function for Y defined as in the proof of theorem 1. Then ρ is a semimetric for Y, since any almost-open mapping is pseudo-open.

To show that each convergent sequence is Cauchy, let $\{y_n\}$ be a asequence in Y which is convergent to y and $\varepsilon>0$ be given.

Since f is almost-open, there exist $x \in f^{-1}y$ and a local base $\mathscr{B}$ of x such that $f(V)$ is open in Y for each $V \in \mathscr{B}$.

On the other hand, since all convergent sequences in X are Cauchy under the semimetric d, it is easy to verify that there exists a positive real number δ such that the diameter of $S_d(x, \delta)$ is less than ε.

We can choose a set B from $\mathscr{B}$ contained in $S_d(x, \delta)$. Then $f(B)$ is an open set containing y with diameter less than ε under the quotient-distance ρ. There is a natural number m such that $y_n \in f(B)$ for all $n>m$. For this m, $\rho(y_i, y_j)<\varepsilon$ if $i, j>m$; that is the sequence $\{y_n\}$ is Cauchy.

From the Ponomarev proposition, we have the following

COROLLARY 3. *A T_1-space Y is developable if and only if Y is an almost-open π-image of a metric space.*

We say that a symmetric space (X, d) satisfies the *weak condition of Cauchy* [4] if a subset F of X is closed whenever there is a positive ε such that $d(x, y) \geqq \varepsilon$ for all distinct $x, y \in F$.

Arhangel'skii [2] and Kofner [5] proved: A T_1-space Y is symmetrizable by a symmetric with weak condition of Cauchy if and only if Y is a quotient π-image of a metric space.

Considering the quotient π-images of developable spaces we have a analoguous result.

THEOREM 4. *A space Y is symmetrizaale by a symmetric with weak condition of Cauchy if and only if Y is a quotient π-image of a developable space.*

Proof. Necessity. Clear from the Arhangel'skii and Kofner's rsuelt cited above.

Sufficiency. Let (X, d) be the semimetric space in which every convergent sequence is Cauchy, $f: X \rightarrow Y$ be a quotient π-mapping and ρ be the quotient-distance function on Y induced by d and f. It is easy to show that ρ is a symmetric for Y.

To show that ρ satisfies the weak condition of Cauchy, let F be a subset of Y, such that $\rho(x, y) \geqq \varepsilon$ for all distinct $x, y \in F$, where ε is some fixed positive real. Thus $d(f^{-1}x, f^{-1}y) \geqq \varepsilon$ for all $x, y \in F$. Since every convergent sepuence is Cauchy under the semimetric d, for any $u \in X$ there exists a positive real $\delta(u, \varepsilon)$ such that $d(u, u_1) < \delta(u, \varepsilon)$ and $d(u, u_2) < \delta(u, \varepsilon)$ imply that $d(u_1, u_2) < \varepsilon$. From these considerations we know that $\{f^{-1}x : x \in F\}$ is a discrete family of clased subsets of X. Thus $f^{-1}F$ is closed in X. Now the quotientness of f implies that F is closed in Y.

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Seoul National University