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ON CLASSIFICATION OF PATHS IN GEOMETRY OF CONNECTION

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1 Introduction

Let M be an n-dimensional differentiable manifold covered by a system of coordinate neighborhood $\{U : x_h\}$ in which a system of paths is given by

$$\frac{d^2x^h}{dt^2}+\Gamma_{jk}{}^h(x)\frac{dx^j}{dt}\frac{dx^k}{dt}=\lambda\frac{dx^h}{dt}.$$

λ being a scalar field, where the indices $h, j, i, \cdots$run over the range $\{1, 2, \cdots, n\}$.

A change of affine connection $\Gamma_{ji}{}^h(=\Gamma_{ij}{}^h)$ which does not change the system of paths is given by

$$\bar{\Gamma}_{ji}{}^h=\Gamma_{ji}{}^h+\delta_j{}^h p_i+\delta_i{}^h p_j,$$

where p_i is an arbitrary covecter field, and is called a projective change of Γ.

We consider an n-dimensional differentiable manifold M, in which two different connections $\Gamma_{ji}{}^h$, $\bar{\Gamma}_{ji}{}^h$ are given.

Let v^h be a parallel vector field, with respect to $\Gamma_{ji}{}^h$ along the path $x^h(t)$ is given by

(1.1) $$\frac{d^2x^h}{dt^2}+\bar{\Gamma}_{jk}{}^h\frac{dx^j}{dt}\frac{dx^k}{dt}=\varphi_1(x)\frac{dx^h}{dt},$$

then v^h satisfies following eqations

(1.2) $$\frac{dv^h}{dt}+\Gamma_{jk}{}^h\frac{dx^j}{dt}v^k=\varphi_2(t)v^h.$$

for some functions φ_1, φ_2.

If there exists a mixed tensor $H_i{}^h$ which is related by the equation

(1.3) $$v^h=H_j{}^h\frac{dx^j}{dt}$$

along the path $x^h(t)$, then we shall call such a change of connection a pseudo projective change related to H.

A pseudo projective change related to H of an affine connection, in general, is given by

(1.4) $$\frac{1}{2}(\bar{\Gamma}_{jk}{}^h+\bar{\Gamma}_{kj}{}^h)=\frac{1}{2}(\Gamma_{jk}{}^h+\Gamma_{kj}{}^h)+u_j\delta_k{}^h+u_k\delta_j{}^h+\frac{1}{2}H_l{}^h(\nabla_j H_k{}^l+\nabla_k H_j{}^l)$$

where u_j is an arbitrary covecter, $\bar{H}_l{}^h H_k{}^l=\delta_k{}^h$ and ∇_j is covariant differentiation with respect to $\Gamma_{jk}{}^h$ [1].

In the change of (1.4), if $H_i{}^h$ is covariantly constant with respect to $\Gamma_{jk}{}^h$, then the change of (1.4) is a projective change of connections in an ordinary way. We can obtain various such changes corresponding to H.

2. Projective change of connection

If we consider that the connection is symmetric, then the pseudo projective change of Γ related to H is given by

(2.1) $$\bar{\Gamma}_{jk}{}^h=\Gamma_{jk}{}^h+u_j\delta_k{}^h+u_k\delta_j{}^h+T_{jk}{}^h$$

where,

(2.2) $$T_{jk}{}^h=\frac{1}{2}H_l{}^h(\nabla_j H_k{}^l+\nabla_k H_j{}^l)$$

By straightforward computation, we can find the curvature tensor of $\bar{\Gamma}_{kj}{}^h$, that is

(2.3) $$\bar{R}_{kji}{}^h=R_{kji}{}^h+\delta_j{}^h u_{ki}-\delta_k{}^h u_{ji}+T_{kji}{}^h$$

where, $$u_{ji}=\nabla_j u_i-u_j u_i-u_t T_{ji}{}^t$$

$$T_{kji}{}^h=\nabla_k T_{ji}{}^h-\nabla_j T_{ki}{}^h+T_{ji}{}^t T_{kt}{}^h-T_{ki}{}^t T_{jt}{}^h.$$

Eliminating u_{ji}, from (2.3), we have

(2.4) $$\bar{P}_{kji}{}^h=P_{kji}{}^h+H_{kji}{}^h$$

where, P and $\bar{P}$ are the projective curvature tensor, and

(2.5) $$H_{kji}{}^h=T_{kji}{}^h+\frac{1}{n-1}\delta_j{}^h T_{ski}{}^s-\frac{1}{n-1}\delta_k{}^h T_{sji}{}^s.$$

Thus, we have next theorem.

THEOREM 1. *If, in an n-dimensional differentiable manifold, there exists a mixed tensor $H_j{}^h$ such that $H_{kji}{}^h=0$, then the projective curvature tensor is invariant under the pseudo projective change related to H.*

3. Pseudo Projective change related to pseudo F-conformal Killing tensor in an almost complex manifold

Let C^n be an n-dimensional almost complex manifold with a Riemannian metric g_{ji}, and with an almost complex structure F_i^h that is;

(3.1) $$F_i^l F_l^h = -\delta_i^h,\quad F_j^l F_k^t g_{lt} = g_{jk},\quad F_{jk} = F_j^l g_{lk} = -F_{kj}.$$

If C^h is a symmetric conformally flat space, then we can take a structure tensor F_i^h which is a pseudo F-conformal Killing tensor defined by

(3.2) $$\nabla_j F_k^h = q^h g_{jk} - q_k \delta_j^h + p_k F_j^h - p^h F_{jk},$$

where $p_i = \partial_i p$, $q_k = p_l F_k^l$ and p is an arvitrary scalar function [2]. Such a connection Γ, we shall call a conformally flat symmetric F-connection. In an almost complex manifold with a conformally flat symetric F-connection, if we put $H_j^h = F_j^h$, since $\bar{H}_j^h = -F_j^h$, then we have a pseudo projective change related to F. This change is given by

(3.3) $$\frac{1}{2}(\bar{\Gamma}_{jk}^h + \bar{\Gamma}_{kj}^h) = \Gamma_{jk}^h + \left(u_j + \frac{1}{2}p_j\right)\delta_k^h + \left(u_k + \frac{1}{2}p_k\right)\delta_j^h - p^h g_{jk} + \frac{1}{2}q_j F_k^h + \frac{1}{2}q_k F_j^h.$$

Since u_j is an arbitrary covector, we can take $u_j = \frac{1}{2}p_j$ and if we put

$$\bar{\Gamma}_{jk}^h = \bar{\Gamma}_{kj}^h$$

then we have

(3.4) $$\bar{\Gamma}_{jk}^h = \Gamma_{jk}^h + p_j\delta^h_k + p_k\delta_j^h - p^h g_{jk} + \frac{1}{2}F_k^h q_j + \frac{1}{2}F_j^h q_k.$$

By a straightforward computation, we can find the curvature tensor of $\bar{\Gamma}_{jk}^h$, that is,

$$\bar{R}_{ijk}^h = R_{ijk}^h - \delta_i^h\left(p_{jk} - \frac{1}{4}q_j q_k\right) + \delta_j^h\left(p_{ik} - \frac{1}{4}q_i q_k\right) - g_{jk}\left(p_i^h - \frac{1}{4}q_i q^h\right) + g_{ik}\left(p_j^h - \frac{1}{4}q_j q^h\right) + \frac{1}{2}F_j q_{ik}^h - \frac{1}{2}F_i^h q_{jk} + \frac{1}{2}F_k^h(q_{ij} - q_{ji})$$

where

$$p_{ij} = \nabla_i p_j - p_i p_j + \frac{1}{2}p_t p^t g_{ij}$$

$$q_{ij} = \nabla_i q_j - \frac{1}{2}q_i p_j - \frac{1}{2}p_i q_j$$

Since the manifold is conformally flat, we have

(3.6) $$\bar{R}_{ijk}^h = \frac{1}{4}\delta_i^h q_j q_k - \frac{1}{4}\delta_j^h q_i q_k + \frac{1}{2}F_j^h q_{ik} - \frac{1}{2}F_i^h q_{jk} + \frac{1}{2}F_k^h(q_{ij} - q_{ji})$$

We denote R instead $\bar{R}$ in (3.6), and eliminating q, we have

$$(3.7)\qquad C_{kji}{}^{h}=R_{kji}{}^{h}-\frac{n-1}{n(n-2)}(P_{ki}F_{j}{}^{h}-P_{ji}F_{k}{}^{h}+R_{ji}\delta_{k}{}^{h}-R_{ki}\delta_{j}{}^{h})$$
$$+\frac{1}{n(n-2)}(P_{tj}F^{t}{}_{i}\delta_{k}{}^{h}-P_{tk}F^{t}{}_{i}\delta_{j}{}^{h}+H_{ij}F_{k}{}^{h}-H_{ik}F_{j}{}^{h})$$
$$+\frac{1}{n(n+2)(n-2)}(Q_{ik}F_{j}{}^{h}-Q_{ij}F_{k}{}^{h}+Q_{jt}F^{t}{}_{i}\delta_{k}{}^{h}-Q_{kt}F^{t}{}_{i}\delta_{j}{}^{h})$$
$$-\frac{1}{(n+2)(n-2)}Q_{jk}F_{i}{}^{h})=0,$$

where,

$$P_{ki}=R_{ski}{}^{t}F_{t}{}^{s},\quad H_{ji}=R_{js}F_{i}{}^{s}$$

and

$$Q_{ji}=H_{ij}-H_{ji}-(n-1)R_{ijs}{}^{t}F_{t}{}^{s}.$$

References

[1] O. Yoon, *On extended projective change of connections* J. Korean Math. Soc., Vol. 10, No. 2, pp. 89~91(1973).

[2] O. Yoon, *On conformal Killing teosors in a Riemannian manifold*, J. Korean Math. Soc. Vol. 10, No. 2, pp. 85~87 (1973).

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