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NOTE ON UMBILICAL HYPERSURFACES WITH UNIT VECTOR FIELDS OF A REAL SPACE FORM

BY U-HANG KI AND SOON HWA ANN

Introduction.

Recently, Lawson [2] has studied a hypersurface in a real space form of constant mean curvature which has parallel Ricci tensor. With use of these results, Mogi and Nakagawa [5] have given a classification of hypersurface in a real space form with parallel Ricci tensor or the Cartan's condition about Ricci tensor.

In the present paper, we consider umbilical hypersurface M with unit vector fields in a real space form $\bar{M}(c)$, that is, there exist mutually orthogonal unit vector fields U and V such that the second fundamental tensor H of M with induced Riemannian metric tensor g has the from

$$H=\alpha I+\beta(u\otimes U+v\otimes V),$$
$$g(U,X)=u(X),\quad g(V,X)=v(X)$$

for any vector field X, α and β being functions on M.

First of all we shall prepare some local properties about a hypersurface of a real space form. In the last section 2, we prove some lemmas on an umbilical hypersurface with unit vector fields, and give classifications of the space.

§1. Certain hypersurfaces of a real space form.

Let $\bar{M}(c)$ be an $(n+1)$-dimensional real space form covered by a system of coordinate neighborhoods $\{\bar{U};y^\kappa\}$, where here and in this section the indices $\lambda,\mu,\nu,\kappa,\cdots$ run over the range $\{1,2,3,\cdots,n+1\}$, that is, the curvature tensor of $\bar{M}(c)$ has the form

$$K_{\nu\mu\lambda\kappa}=c(g_{\lambda\mu}g_{\nu\kappa}-g_{\nu\lambda}g_{\mu\kappa}), \tag{1.1}$$

c being constant, where $g_{\lambda\mu}$ are components of Riemannian metric tensor of $\bar{M}(c)$.

Let M be an n-dimensional hypersurface which is covered by a system of coordinate neighborhoods $\{U; x^h\}$, where here and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1, 2, 3, \cdots, n\}$, and which is differentially immersed in $\bar{M}(c)$ by $X: M \to \bar{M}$, i.e., $y^\kappa = y^\kappa(x^h)$.

We put $B_i{}^\kappa = \partial y^\kappa / \partial x^i$, $\partial_i = \partial / \partial x^i$, then componenents g_{ji} of the induced metric tensor of M are given by $g_{ji} = g_{\lambda\mu} B_j{}^\lambda B_i{}^\mu$. $B_i{}^\kappa$ are, for each i, local vector fields of $\bar{M}(c)$ tangent to M and the vectors $B_i{}^\kappa$ are linearly independent in each coordinate neighborhood. $B_i{}^\kappa$ is, for each κ, a local 1-form of M.

We choose a unit vector C^κ of $\bar{M}$ normal to M in such a way that $n+1$ vectors $B_i{}^\kappa$, C^κ give the positive orientation of $\bar{M}$.

We denote $\{{}_j{}^h{}_i\}$ and ∇_i by the Christoffel symbols formed with Riemannian metric g_{ji} and the operator of covariant differentiation with respect to $\{{}_j{}^h{}_i\}$ respectively. Then the equations of Gauss and Weingarten are respectively

$$\nabla_j B_i{}^\kappa = \partial_j B_i{}^\kappa + \{{}_\mu{}^\kappa{}_\lambda\} B_j{}^\mu B_i{}^\lambda - B_h{}^\kappa \{{}_j{}^h{}_i\} = h_{ji} C^\kappa, \tag{1.2}$$

$$\nabla_j C^\kappa = \partial_j C^\kappa + \{{}_\mu{}^\kappa{}_\lambda\} B_j{}^\mu C^\lambda = -h_j{}^t B_t{}^\kappa, \tag{1.3}$$

where h_{ji} are the components of second fundamental tensor with respect to the normal C^κ, $h_j{}^h$ defined by $h_j{}^h = h_{jt} g^{th}$ and $(g^{ji}) = (g_{ji})^{-1}$.

In the sequel, we need the structure equations of the hypersurface M, that is, the following equations of Gauss

$$K_{kjih} = c(g_{kh} g_{ji} - g_{jh} g_{ki}) + h_{kh} h_{ji} - h_{jh} h_{ki}, \tag{1.4}$$

where K_{kjih} are covariant components of the curvature tensor of M, and equations of Codazzi,

$$\nabla_k h_{ji} - \nabla_j h_{kj} = 0. \tag{1.5}$$

From equations (1.4) of Gauss, we have the relationships

$$K_{ji} = (n-1) c g_{ji} + (h_t{}^t) h_{ji} - h_{jt} h_i{}^t \tag{1.6}$$

and hence

$$K = n(n-1)c + (h_t{}^t)^2 - h_{ji} h^{ji}, \tag{1.7}$$

where K_{ji} and K are respectively the components of Ricci tensor and the curvature scalar of M.

A hypersurface M of dimension n is said to be an umbilical form with unit vector fields, if there exist on M, two mutually orthogonal unit vector fields u^h and v^h such that

$$h_{ji}=\alpha g_{ji}+\beta(u_ju_i+v_jv_i) \tag{1.8}$$

for some functions α and β.

From the relation above, we find

$$h_t^{\ t}=n\alpha+2\beta, \tag{1.9}$$

$$h_{jt}u^t=(\alpha+\beta)u_j, \qquad h_{jt}v^t=(\alpha+\beta)v_j, \tag{1.10}$$

$$h_{ji}h^{ji}=n\alpha^2+4\alpha\beta+2\beta^2 \tag{1.11}$$

because u^h and v^h are unit orthogonal. Thus the second fundamental tensor $(h_j^{\ h})$ has at most two eigenvalues α and $\alpha+\beta$ of multiplications n-2 and 2 respectively.

If we substitute (1.10) and (1.11) into (1.7), we get

$$K=n(n-1)(c+\alpha^2)+4(n-1)\alpha\beta+2\beta^2. \tag{1.12}$$

§ 2. Umbilical hypersurface with unit vector fields.

Throughout this paper we consider the hypersurface M of dimension $n>3$ is an umbilical form with unit vector fields.

LEMMA 2.1. *Let M be an umbilical form with unit vector fields of dimension $n>3$ such that the curvature scalar K is constant. Then α and β are constants on M.*

Proof. Differentiating (1.8) covariantly along M, we have

$$\begin{aligned}\nabla_k h_{ji}=&\alpha_k g_{ji}+\beta_k(u_ju_i+v_jv_i)\\ &+\beta\{(\nabla_k u_j)u_i+(\nabla_k u_i)u_j+(\nabla_k v_j)v_i+(\nabla_k v_i)v_j\},\end{aligned} \tag{2.1}$$

from which, taking skew-symmetric parts with respect to k and j and using (1.5),

$$\begin{aligned}&\alpha_k g_{ji}-\alpha_j g_{ki}+\beta_k(u_ju_i+v_jv_i)-\beta_j(u_ku_i+v_kv_i)\\ &+\beta\{(\nabla_k u_j-\nabla_j u_k)u_i+(\nabla_k u_i)u_j-(\nabla_j u_i)u_k\\ &+(\nabla_k v_j-\nabla_j v_k)v_i+(\nabla_k v_i)v_j-(\nabla_j v_i)v_k\}=0,\end{aligned} \tag{2.2}$$

where $\nabla_k\alpha$ is denoted by α_k. If we transvect (2.2) with $u^jv^iv^k$ and $u^iv^ju^k$, we have respectively

(2.3) $$u^t\alpha_t+u^t\beta_t=0,$$

(2.4) $$v^t\alpha_t+v^t\beta_t=0$$

because u^h and v^h are unit orthogonal.

Differentiating (1.12) covariantly, we find

(2.5) $$(n-1)(n\alpha+2\beta)\alpha_j+2\{(n-1)\alpha+\beta\}\beta_j=0.$$

by virtue of K=constant, from which, transvecting u^j and using (2.3),

(2.6) $$(n-2)\{(n-1)\alpha+2\beta\}(\alpha_t u^t)=0.$$

If $\alpha_t u^t\neq 0$, then $(n-1)\alpha+2\beta=0$ which implies $(n-3)\nabla_j(\alpha^2)=0$. This contradict $\alpha_t u^t\neq 0$. Consequently we have

(2.7) $$\alpha_t u^t=0, \quad \beta_t u^t=0.$$

In the same way we also have from (2.4)

(2.8) $$\alpha_t v^t=0, \quad \beta_t v^t=0.$$

Next, transvecting (2.2) with g^{ji} and taking account of (2.7) and (2.8), we obtain

(2.9) $$(n-1)\alpha_k+2\beta_k$$
$$=\beta\{u^t\nabla_t u_k+v^t\nabla_t v_k+(\nabla_t u^t)u_k+(\nabla_t v^t)v_k\}.$$

On the other hand, if we transvect (2.2) with $u^j u^i$ and $v^j v^i$, we get respectively

(2.10) $$\alpha_k+\beta_k+\beta\{-u^t\nabla_t u_k-(u^s u^t\nabla_t v_s)v_k\}=0,$$

(2.11) $$\alpha_k+\beta_k+\beta\{-v^t\nabla_t v_k-(v^s v^t\nabla_t u_s)u_k\}=0.$$

Combining (2.9), (2.10) and (2.11), we conclude

(2.12) $$(n-3)\alpha_k=-\beta\{(v^s v^t\nabla_t u_s-\nabla_t u^t)u_k+(u^s u^t\nabla_t v_s-\nabla_t v^t)v_k\},$$

which implies that $v^s v^t\nabla_t u_s=\nabla_t u^t$, $u^s u^t\nabla_t v_s=\nabla_t v^t$ because of (2.7) and (2.8). Thus (2.12) means α is constant for $n>3$ and hence β is also by virtue of (2.5). Therefore, Lemma 2.1 is proved.

LEMMA 2.2. *Under the same assumptions as those stated in Lemma 2.1 we have* $\nabla_k h_{ji}=0$ *and consequently* $\nabla_k K_{ji}=0$.

Proof. α and β being constants because of Lemma 2.1, we see from (2.2) that $\beta=0$ or

$$(\nabla_k u_j-\nabla_j u_k)u_i+(\nabla_k u_i)u_j-(\nabla_j u_i)u_k + (\nabla_k v_j-\nabla_j v_k)v_i+(\nabla_k v_i)v_j-(\nabla_j v_i)v_k=0. \tag{2.13}$$

If $\beta=0$, then M is totally umbilical by virtue of (1.8) and hence $\nabla_k h_{ji}=0$. Thus we may only consider $\beta\neq 0$.

Transvecting (2.13) with u^i and v^i, we obtain respectively

$$\nabla_k u_j-\nabla_j u_k=A_j v_k-A_k v_j, \tag{2.14}$$

$$\nabla_k v_j-\nabla_j v_k=A_k u_j-A_j u_k, \tag{2.15}$$

where $A_j=u^t\nabla_j v_t$.

From (2.14) and (2.15) we have

$$v^t\nabla_t u_j=-(v^tA_t)v_j, \qquad u^t\nabla_t u_j=-(u^tA_t)v_j, \tag{2.16}$$

$$u^t\nabla_t v_j=(u^tA_t)u_j, \qquad v^t\nabla_t v_j=(v^tA_t)u_j. \tag{2.17}$$

Substituting (2.14) and (2.15) into (2.13), we obtain

$$A_k(u_jv_i-v_ju_i)+A_j(v_ku_i-u_kv_i) +u_j\nabla_k u_i-u_k\nabla_j u_i+v_j\nabla_k v_i-v_k\nabla_j v_i=0. \tag{2.18}$$

Transvecting (2.18) with u^j, v^j and taking account of (2.16) and (2.17), we find respectively

$$\nabla_k u_i=-A_k v_i, \tag{2.19}$$

$$\nabla_k v_i=A_k u_i. \tag{2.20}$$

Thus (2.1) implies $\nabla_k h_{ji}=0$ because of (2.19), (2.20) and Lemma 2.1. Thus (1.6) proves the last assertion of the lemma.

LEMMA 2.3. *Under the same assumptions as those stated in Lemma* 2.1, *we have*

$$\alpha(\alpha+\beta)+c=0. \tag{2.21}$$

Proof. Differentiating (2.19) covariantly and using (2.20), we get

$$\nabla_k\nabla_j u_i=-(\nabla_k A_j)v_i-A_jA_ku_i, \tag{2.22}$$

from which, taking skew-symmetric parts with respect to k and j and making use of the Ricci identity, (2.23) $$-K_{kjih}u^h=(\nabla_jA_k-\nabla_kA_j)v_i,$$

or, using (1.4) and (1.10),

(2.24) $$(\nabla_kA_j-\nabla_jA_k)v_i =c(u_kg_{ji}-u_jg_{ki})+(\alpha+\beta)(u_kh_{ji}-u_jh_{ki}).$$

Transvecting (2.24) with $u^kv^jv^i$, $g^{ji}u^k$ and using (1.9) and (1.10), we have respectively

$$u^tv^s\nabla_tA_s-u^tv^s\nabla_sA_t=c+(\alpha+\beta)^2,$$
$$u^tv^s\nabla_tA_s-u^tv^s\nabla_sA_t=(n-1)c+(\alpha+\beta)\{(n\alpha+2\beta\}-(\alpha+\beta)\}.$$

The last two relations imply (2.21). This completes the proof of the lemma.

In the case where ambient space $\tilde{M}$ is Euclidean, from (1.12) and (2.21), we have

(2.25) $$K=(n-1)(n-4)\alpha^2+2\beta^2\geqq 0.$$

If the curvature scalar K is positive, by completeness, M is congruent to $S^2(r)\times E^{n-2}$ or $S^{n-2}(r)\times E^2$, and if $K=0$, M is cylindrical because the Ricci tensor is parallel (cf. [3], [5]).

Thus we have proved

THEOREM 2.4 *Let M be a complete and connected umbilical hypersurface with unit vector fields defined by (1.8) such that dim $M>3$ and the curvature scalar K is constant. Then M is congruent to $S^2(r)\times E^{n-2}$ or $S^{n-2}(r)\times E^2$ if the scalar curvature $K>0$, and M is a cylinder if the scalar curvature $K=0$.*

Now, we suppose that the real space form $\tilde{M}(c)$ is of constant curvature $c\neq 0$ and the hypersurface M has the constant scalar curvature K and $n>3$. Then by means of Lemma 2.1, 2.2 and 2.3, we have two cases: (1) M has exactly two distinct constant principal curvatures, say α and $\alpha+\beta$ of multiplicities $n-2$ and 2 respectively, such that $c+\alpha(\alpha+\beta)=0$, and (2) M is totally umbilic but not totally geodesic.

For the first case, we use Lemma 2.3. Then, from the straightforward argument used by Lawson [2], we obtain the following conclusion:

If $c>0$, then M is isometric to $S^2(c_1)\times S^{n-2}(c_2)$, and if $c<0$, then M is isometric to $S^2(c_1)\times H^{n-2}(c_2)$, $S^r(a)$ being a sphere with curvature c and $H^r(a)$ a hyperbolic space with curvature a.

For the second case, M is totally umbilic but not totally geodesic. If $c>0$, then M is isometric to a sphere S^n, and if $c<0$, then M is a sphere S^n, a hyperbolic space H^n whose curvature is different from c, or a flat hypersurface F^n.

Thus, summing up the results obtained above, we have proved

THEOREM 2.5. *Let $\bar{M}$ be an $(n+1)$-dimensional and simply connected real space form with curvature $c\neq 0$ and let $M(n>3)$ be a complete and connected umbilical hypersurface with unit vector fields defined in* (1.8) *such that the curvature scalar K is constant. Then the following statements are true:*

(1) *If $c>0$, then M is isometric tothe great sphere, the small sphere or $S^2(c_1)\times S^{n-2}(c_2)$, where $1/c_1+1/c_2=1/c$.*

(2) *If $c<0$, then M is isometric to S^n, H^n, F^n or $S^2(c_1)\times H^{n-2}(c_2)$, $S^{n-2}(c_1)\times H^2(c_2)$, where $1/c_1+1/c_2=1/c$.*

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Kyungpook University