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THE PROBABILITY OF WINNING A TABLE TENNIS GAME

By H. S. HAHN

1. Introduction.

Table tennis has lately become an internationally popular competitive sport with the majority of its players living in the Far East. This note is an attempt to analyze mathematical aspects of the game under a few simple assumptions. Consider two players A and B. Player A wins a game if he reaches 21 points or more with at least a 2 point spread over his opponent B. Let (i, j) denote a pair of integers, which represent the scores of A and B, respectively, and denote as "deuce" all scores of the form (i, i), $i>19$, as "Add A" all scores of the form $(i+1, i)$, $i>19$, and as "Game A" all scores of the form either $(21, j)$, $j<20$ or $(i+2, i)$, $i>19$, etc. We note that besides these 5 scores (deuce, Add A, Add B, Game A and Game B) there are 440 possible scores of the form (i, j), $i\leq 20$, $j\leq 20$ and $i+j<40$. We assume that Player A has the independent probability, p, of winning any given point, say from a score (i, j) to $(i+i, j)$, and Player B has the probability $q(\leq p)$ with $p+q=1$. We classify games into two kinds: one terminating without a deuce, called a straight game, and the other involving one or more deuces, called a deuce game (a deuce game can be indefinitely long).

2. A straight game.

The probability that Player A wins in a straight game is

$$P_A \text{ (straight)} = p^{21}\sum_{i=0}^{19}\binom{20+i}{i}q^i,$$

where the probability with the end score (21, i) is

$$P(21, i) = \binom{20+i}{i}p^{21}q^i,$$

for there are $\binom{20+i}{i}$ permutations of arranging 21 p's and i q's with a p at the end. The table below shows this probability for two interesting values of p, $p=.6$ and $p=.51$: (to 4 places)

	$p=.6$		$p=.51$	
$P(21, 0)=$	.0000		.0000	
$P(21, 1)=$	.0002		.0000	
$P(21, 2)=$	.0008		.0000	
$P(21, 3)=$	.0025		.0002	
$P(21, 4)=$	.0060		.0004	
$P(21, 5)=$	.0120		.0011	
$P(21, 6)=$	.0207		.0023	
$P(21, 7)=$	.0319		.0044	
$P(21, 8)=$	.0447		.0075	
$P(21, 9)=$	.0576		.0118	
$P(21, 10)=$	.0691		.0173	
$P(21, 11)=$	.0779		.0239	
$P(21, 12)=$	.0831		.0313	
$P(21, 13)=$	.0844	(Max.)	.0389	
$P(21, 14)=$	.0820		.0463	
$P(21, 15)=$	.0765		.0529	
$P(21, 16)=$	.0689		.0583	
$P(21, 17)=$	.0599		.0622	
$P(21, 18)=$	.0506		.0644	
$P(21, 19)=$	.0416		.0647	(Max.)
P_A (straight) $=$	.8702		.4878	

This table shows that if there is a significant difference between p and q, such as $p=.6$ and $q=.4$, the game is most likely to end (with probability close to 90%) in straight with an end game score around (21, 13), i. e., with the mode of length of game 34 points. But if the difference is slight, such as $p=.51$ and $q=.49$, the game will be extended to a deuce game with better than fifty-fifty chance.

3. A deuce game.

The analysis of a straight game is straightforward and its probability is easily computed. However, a deuce game is of a complex nature, for it is the problem of a random walk, called a Markov chain, with two absorbing states. Denote the five states Game A, Add A, deuce, Add B, and Game B as s_0, s_1, s_2, s_3 and s_4, resrectively, where s_0 and s_4 are absorbing states and s_1, s_2, s_3, are transient states. Consider a 3×3 matrix $Q=(q_{ij})$, where q_{ij} is the probability of going from one transient state s_i to another s_j in one step, and a 3×2 matrix $R=(r_{ik})$, where r_{ik} is the probability of going from a transient state s_i to an absorbing state s_k. Hence we have

$$Q=\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} 0 & q & 0 \\ p & 0 & q \\ 0 & p & 0 \end{pmatrix} \end{array}, \qquad R=\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \end{array}\!\!\begin{array}{c} \begin{array}{cc} s_0 & s_4 \end{array} \\ \begin{pmatrix} p & 0 \\ 0 & 0 \\ 0 & q \end{pmatrix} \end{array}.$$

Let $P=(p_{ik})$ be a 3×2 probability matrix, where p_{ik} is the probability of starting at a transient state s_i and ending in an absorbing state s_k in *one step or more*. Then we should have

$$p_{ik}=r_{ik}+\textstyle\sum_{r=1}^{3} q_{it}p_{tk}.$$

This implies $P=R+QP$ or $(I-Q)P=R$. Since $\det(I-Q)=1-2pq\neq 0$, $(I-Q)^{-1}$ exists, and therefore $P=(I-Q)^{-1}R$, or

$$\begin{pmatrix} p_{10} & p_{14} \\ p_{20} & p_{24} \\ p_{30} & p_{34} \end{pmatrix} = (p^2+p^2)^{-1}\begin{pmatrix} p(1-pq) & q^3 \\ p^2 & q^2 \\ p^3 & q(1-pq) \end{pmatrix}.$$

Thus we get

$$p_{20}=p^2/(p^2+q^2)$$

and this is all we need in computing the probability that A wins a deuce game. We see that

$$P_A(\text{deuce game})=P(20,20)p_{20}=\binom{40}{20}p^{20}\ q^{20}\cdot p^2/(p^2+q^2)$$

$$=\begin{cases} .03836 \text{ if } p=.6, \\ .06467 \text{ if } p=.51. \end{cases}$$

4. The probability of Game A.

From sections 2 and 3 above we obtain

$$P_A=P_A(\text{straight game})+P_A(\text{deuce game})$$

$$=p^{21}\ \textstyle\sum_{i=0}^{91}\binom{20+i}{i}\ q^i+\binom{40}{20}p^{22}q^{20}/(p^2+q^2)$$

$$=\begin{cases} .9085 \text{ if } p=.6 \\ .5525 \text{ if } p=.51. \end{cases}$$

The case $p=.6$ shows that the difference between p and q is magnified by the length of a game to greatly increase A's chance of winning. The case $p=.51$, however, shows that such magnification is too slight to predict a winner.

REMARK. In a real game the server has usually a clear advantage and therefore a more realistic approach is to distinguish between two probabilities, p_1, of A winning a point when serving, and p_2 when receiving, with corresponding probabilities q_1 and q_2 for B. Then

$$P(21, 0)=p_1^{11}p_2^{10} \text{ or } p_1^{10}p_2^{11}$$

and

$$P(21, 1)=11p_1^{10}q_1p_2^{10}+10p_1^{11}p_2^{9}q_2$$

or

$$11p_2^{10}q_2p_1^{10}+10p_2^{11}p_1^{9}q_1,$$

$\cdots$, etc., depending on whether A serves or receives first. The reader should persue this estimation if our simple approach is not satisfactory.

Reference

J. G. Kemeny and J. L. Snell, *Finite Markov Chains, Princeton,* N. J., Van Nostrand, 1960.

West Georgia College