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L^1-LIPSCHITZ CONDITION AND THE EXISTENCE OF SOLUTIONS IN $L^1(0,1)$

BY JONGSIK KIM

One of the standard methods of proving the theorem that the Lipschitz condition guarantees the existence of the unique solution of $y'=f(x,y)$ with an initial condition is that of using the contraction mapping (cf. [1]). In this paper, we shall make a straight-forward generalization of the above method to the space of Lebesgue integrable functions and get a similar result.

We recall that when $X=(X,\rho)$ is a metric space, a mapping $T: X\to X$ is called a *contraction mapping* in X if there is a number k, with $0<k<1$, such that $x,y\in X$, $x\neq y$, implies

$$\rho(Tx,\ Ty)\leqq k\rho(x,y).$$

We shall use the following definition.

DEFINITION 1. Let $I=(0,1)$ and R be the set of real numbers. Let $f(x,y)$ be a function on $I\times R$. We shall say that $f(x,y)$ satisfies *L^1-Lipschitz condition* in $L^1(0,1)$ if $f(x,g(x))\in L^1(0,1)$ for any $g(x)\in L^1(0,1)$ and also if for every $g_1(x)$ and $g_2(x)$ in $L^1(0,1)$

$$\| f(x,g_1(x))-f(x,\ g_2(x))\| \leqq k\| g_1(x)-g_2(x)\|$$

with $0<k<1$.

We note that if g is a continuous real valued function on $(0,1)$ with

$$\sup_{x\in(0,1)}|g(x)|<1,$$

then $f(x,y)=g(x)y$ trivially satisfies L^1-Lipschitz condition. It can be shown that $f(x,y)=\frac{1}{2}(1+x|y|)^{1/2}$ satisfies also L^1-Lipschitz condition.

We shall use the following well-known theorem without proof (cf. [1]).

THEOREM 1. *If X is a complete metric space and T is a contraction mapping, then T has a unique fixed point.*

The following theorem is a direct result following from the definition of L^1-Lipschitz condition.

THEOREM 2. *If $f(x,y)$ satisfies L^1-Lipschitz condition in $L^1(0,1)$, then the mapping $T:L^1(0,1) \to L^1(0,1)$ defined as $T(g(x))=f(x,g(x))$ for any $g(x)$ in $L^1(0,1)$ is continuous.*

We shall prove that the equation

$$\frac{dy}{dx}=f(x,y) \text{ a. e.,} \qquad y_0=y(x_0)$$

has a unique absolutely-continuous solution in $L^1(0,1)$ if $f(x,y)$ satisfies L^1-Lipschitz condition.

THEOREM 3. *If $f(x,y)$ satisfies L^1-Lipschitz condition, then for every (x_0,y_0) in $I\times R$, there exists a unique integrable function $y=g(x)$ such that*

(i) *$g(x)$ is absolutely continuous, and*

(ii) *$\frac{dy}{dx}=f(x,y)$ a.e. with $y_0=g(x_0)$.*

We shall first prove the following lemma.

LEMMA 4, *Let $f(x,y)$ be a function on $I\times R$ such that $f(x,g(x))\in L^1(0,1)$ for every $g(x)\in L^1(0,1)$. Then for (x_0,y_0) in $I\times R$, the equation*

$$\frac{dy}{dx}=f(x,y) \text{ a.e.,} \quad y_0=y(x_0)$$

has an absolutely continuous solution in $L^1(0,1)$ if and only if

$$g(x)=g(x_0)+\int_{x_0}^{x}(t,g(t))dt$$

has a solution $g(x)$ in $L^1(0,1)$.

Proof. If $y=g(x)$ is a solution of the given differential equation which is absolutely continuous, then due to 8.19 and 8.21 in [2],

$$g(x)=g(x_0)+\int_{x_0}^{x}g'(t)dt=y_0+\int_{x_0}^{x}f(t,g(t))dt.$$

Conversely if $g(x)$ is a solution of the integral equation, then, since $f(x,g(x))\in L^1(0,1)$, due to 8.17 in [2] $g(x)$ is absolutely continuous and

$$\frac{dg(x)}{dx}=f(x,g(x)) \qquad \text{a.e. with } y_0=g(x_0).$$

Proof of theorem 3.

Consider the mapping defined on $L^1(0,1)$ such that

$$(Tg)(x)=y_0+\int_{x_0}^{x}f(t,g(t))dt$$

for every $g(x)$ in $L^1(0,1)$ and x in $(0,1)$.

Then for any $g_1(x)$ and $g_2(x)$ in $L^1(0,1)$

$$\begin{aligned}
&\| (Tg_1)(x) - (Tg_2)(x) \| \\
&= \int_0^1 \left| \int_{x_0}^{x} \{f(t, g_1(t)) - f(t, g_2(t))\}\, dt \right| dx \\
&\leqq \int_0^1 \int_{x_0}^{x} |f(t, g_1(t)) - f(t, g_2(t))|\, dt\, dx \\
&\leqq \int_0^1 \int_0^1 |f(t, g_1(t)) - f(t, g_2(t))|\, dt\, dx \\
&= \| f(t, g_1(t)) - f(t, g_2(t)) \| \\
&\leqq k \| g_1(t) - g_2(t) \|
\end{aligned}$$

Therefore T is a contraction mapping. By theorem 1, there exists $y = g(x)$ in $L^1(0, 1)$ such that $(Tg)(x) = g(x)$ in $L^1(0,1)$. That is,

$$g(x) = y_0 + \int_{x_0}^{x} f(t, g(t))\, dt \qquad \text{a. e.}$$

Since Tg is absolutely continuous, we may choose $g(x)$ to be absolutely continuous. Then

$$g(x) = y_0 + \int_{x_0}^{x} f(t, g(t))\, dt.$$

By the lemma, we obtain the theorem.

References

[1] C. Goffman and H. Pedrick; *First course in functional analysis*, Prentice Hall., 1966.

[2] W. Rudin; *Real and complex analysis*, McGraw Hill, 1972.

Seoul National University