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CHAIN CONDITIONS AND Q-MODULES

BY WUHAN LEE

It will be assumed that all rings have an identity and that the modules are unital. Modules will be right R-modules, and homomorphisms will be R-homomorphisms unless otherwise stated. Previously the author defined a q-module to be an injective module in which every submodule is quasi-injective and obtained several characterizations of a q-modules and investigated the endomorphism ring of a q-module [4].

Later E. Lee [5] established that a left S-submodule ${}_SN$ of M_R, $S=\mathrm{Hom}_R(N, N)$ is noetherian if and only if N_R is noetherian with respect to annihilator submoduels for subsets in R. Further, he studied that if R is a right artinian ring and N_R a submodule of a q-module M_R then the left S-module ${}_SN$ is noetherian. Now the purpose of this paper is to study properties of a q-module with chain conditions over a commutative ring.

Let R be any ring (not necessarily commutative) and M a right R-module. Put $S=\mathrm{Hom}_R(M, M)$, then we assume that M is a left S-module. Let N be a subset of M. Then we denote the annihilator ideal of N in S and in R by $l(N)$ and $\mathrm{ann}\, N$, respectively. Similarly, by $r(A)$ we denote the annihilator submodule of M for a left ideal A in S. We call M a weakly distinguished R-module if for any R-submodules $N_1 \supset N_2$ in M such that N_1/N_2 is R-irreducible, $\mathrm{Hom}_R(N_1/N_2, M)=0$. If M is quasi-injective then M is weakly distinguished if and only if $rl(N)=N$ for any R-submodule N in M [2, Proposition 6].

Finally, we shall assume that a ring R is commutative. In his paper [1] Harada states that if R is a commutative ring and M is a noetherian quasi-injective module then $S=\mathrm{Hom}_R(M, M)$ is left and right artinian. Since every submodule of a q-module is quasi-injective and injectivity of M_R implies that of quasi-injectivity, the following statement is immediate.

PROPOSITION 1. *Let R be a commutative ring and M_R is a noetherian q-module.*

If N is a submodule of M then $S=\mathrm{Hom}_R(N, N)$ is left and right artinian.

Proof. Since every submodule of a notherian module is again noetherian the result is evident by [2, Theorem 1].

Let P be a prime ideal in a commutative ring R. And let $E(R/P)=E$ be an injective hull of R/P. Then Matlis showed in [7] that $E=\bigcup_i A_i$ and $\mathrm{Hom}_R(E, E)$ is a complete local noetherian ring, where $A_j=\{X\in E \mid xP^i=0\}$. Let $\{P_i\}$ be a finite set of distinct maximal ideals in R. Then according to Harada [1], every R-submodule N of $\Sigma\oplus E\ (R/P_i)$ is weakly distinguished and quasi-injective.

Since its implication seems to be interesting, we furnish a rough proof here.

Proof. We may assume that N is an essential submodule of $E=\Sigma\oplus E_i$, $E_i=E(R/P_i)$. Then ann $x\supset \Pi P_i^n$ for any x in N. Let N_1, N_2 be R-submodules of N such that N_1/N_2 is R-irreducible, then $N_1/N_2\approx R/P_i$ for some P_i. Since $N\cap R/P_i\neq(0)$, $\mathrm{Hom}_R(N_1/N_2, N)\neq(0)$, which means that N is weakly distinguished. Hence, E is an R-weakly distinguished injective module. Moreover, if we put $S=\mathrm{Hom}_R(E, E)$, then $S=\mathrm{Hom}_R(E, E)$. Hence, every R-submodule N is an S-submodule by [1, Lemma 1]. Let E' be an injective hull of N contained in E. Then $E=E'\oplus E''$ and $E'\supset N$. $S'=\mathrm{Hom}_R(E', E')$ may be regared as a subring of S. Hence, M is also an S'-module. Therefore, N is R-quasi-injective by [3, Theorem 1. 1].

PROPOSITION 2. *Let R be a commutative noetherian ring and $\{P_i\}$ be a finite set of distinct maximal ideals in R. Then the direct sum of injective hulls $\Sigma\oplus E(R/P_i)$ is a weakly distinguished q-module.*

Proof. The injective hulls are naturally injective and hence the conclusion is immediate from the definition of a q-module.

Now assume that a ring R is not necessary commutative. A. Koehler [4] obtained a charaterization for quasi-injective modules over left artinian rings which have a finitely generated, lower distinguished (contains an isomorphic copy of every simple module), and injective module Q. This class of rings includes quasi-Frobenius rings and finitely generated algebras over commutative artinian rings. According to Koehler, a module M_R over such a ring is quasi-injective if and only if

$$M=\sum_{i=1}^{k}\oplus(\mathrm{Hom}_R(e_iS/e_i\,J, Q))^{g(i)}$$

where $S=\mathrm{Hom}_R(Q,Q)$, e_i is an indecomposable idempotent in S for $i=1, \cdots, k$, J is an ideal of S, the number of nonisomorphic simple R-modules is k, and for $i\neq j$ $e_iS\not\cong e_jS$. This decomposition is unique up to automorphism. Here $\sum\oplus M_i^{g(i)}$ denotes the $g(i)$ copies of M and $g(i)$ can be any cardinal number. If $g(i)=0$, then $M_i^{g(i)}=0$.

PROPOSITION 3. *Let R be a left artinian ring and have a finitely generated lower distinguished, and injective module Q. Then a submodule N_R of a q-module M_R is expressed uniquely (up to automorphism) as*

$$N=\sum_{i=1}^{k}\oplus(\mathrm{Hom}_R(e_iR/e_iJ, R))^{g(i)}$$

where $S=\mathrm{Hom}_R(Q,Q)$, e_i is an indecomposable idempotent in S for $i=1, \cdots, k$, J is an ideal of S, the number of nonisomorphic simple R-modules is k, and for $i\neq j$ $e_iS\not\cong e_jS$.

Proof. Obvious.

COROLLARTY. *Let R be quasi-Frobenius. Then a submodule N_R of a q-module M_R is expressed uniquely (up to automorphism) as*

$$N=\sum_{i=1}^{k}\oplus(\mathrm{Hom}_R(e_iR/e_iJ,\ R))^{g(i)}$$

Proof. R being quasi-Frobenius impies R is left artinian, selfinjective lower distinguished, and finitely generated. Also $R=\mathrm{Hom}_R(R,R)$.

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Seoul National University