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ON A SPACE WITH A QUASI-SEMIREFINED SEQUENCE

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Let X be a topological space, and for each point x in X, let $\mathcal{W}_x$ be a collection of subsets of X which is closed under finite intersections such that each element of $\mathcal{W}_x$ contains x. The collection $\mathcal{W}=\{\mathcal{W}_x:x\in X\}$ is called a *weak base* for X [1], if the following condition holds: A subset V of X is open if and only if for each x in V, there is some W_x in $\mathcal{W}_x$ such that $x\in W_x\subset V$. Now let $(\gamma_1, \gamma_2, \cdots\cdots)$ be a sequence of (not necessarily open) covers of X such that γ_{n+1} is a refinement of γ_n for all $n\in Z^+$. Such a sequence of covers is said to be *semi-refined* [5] if the collection $\{\mathcal{W}_x:x\in X\}$ is a weak base for X, where $\mathcal{W}_x=\text{st}\{(x, \gamma_n):x\in Z^+\}$.

In the definition of semi-refined sequence of covers, we remove the condition of γ_{n+1} being refinement of γ_n and the covering property for the sequence (γ_n), that is, we simply ask that the sequence (γ_n) induces a weak base for X. This leads us to the following

DEFINITION 1. Let $(\gamma_1, \gamma_2, \cdots\cdots)$ be a sequence of collections of subsets of a topological space X. If the family of collections of $\{\mathcal{W}_x:x\in X\}$ is a weak base for X where $\mathcal{W}_x=\{\text{st}(x, \gamma_n):n\in Z^+, x\in \gamma_n^*=\cup\gamma_n\}$, then we say that the sequence is a *quasi-semirefined* (for the brevity qs-refined) *sequence.*

A semi-refined sequence of covers is naturally a qs-refined sequence. From the definition a space with qs-refined sequence is g-first countable [10]. By [7] if a space X is qs-developable it has a qs-refined sequence. For the converse we have the following:

THEOREM 2. *If a topological space X has a qs-refined sequence, and is first countable T_2, then it is qs-developable.*

Proof. Let $(\gamma_1, \gamma_2, \cdots\cdots)$ be a qs-refined sequence of X. It is enough to show that each member of $\mathcal{W}_x$ is a neighborhood of x. There is a decreasing local base at $x \in X$, which we denote by $\{g(n, x) : n \in \boldsymbol{Z}^+\}$. Assume that there is a number $k \in \boldsymbol{Z}^+$ such that $x \in \gamma_k^*$ and $g(n, x)$ is not a subset of $\mathrm{st}(x, \gamma_k)$ for every n. Then there exists a sequence $\langle x_n \rangle$ such that

$$x_n \in g(n, x) - \mathrm{st}(x, \gamma_k)$$

for all n. The sequence $\langle x_n \rangle$ converges to x and $x_n \neq x$ for all $n \in \boldsymbol{Z}^+$. Since X is T_2, the set $X - F$ is open where $F = \{x, x_1, x_2, \cdots\cdots\}$. Now we prove that the set

$$M = (X - F) \cup \{x\}$$

is open. Let $z \in M$ and $z \neq x$. Then $z \in X - F$. Since $X - F$ is open there exists a number $n \in \boldsymbol{Z}^+$ such that $z \in \mathrm{st}(z, \gamma_n) \subset X - F \subset M$. If $z = x$, then $\mathrm{st}(z, \gamma_k)$ is clearly a subset of M. In either case the above argument shows that M is open. This is a contradiction to the fact that $\langle x_n \rangle$ converges to x and that $x_n \neq x$ for each n.

We now take several examples to show that where a space which has a qs-refined sequence is fitted in the classes of spaces.

EXAMPLE 1. There exists a first countable space which does not have a qs-refined sequence. Consider the Sorgenfrey line. If it has a qs-refined sequence, it would be qs-developable by Theorem 1, which is impossible as shown in Example 1 of [7].

For the next example we need the following lemma by Burke. (Theorem 3. 2 of [3]).

LEMMA 3. *A T_1-space X is symmetrizable if and only if it has a semi-refined sequence of covers.*

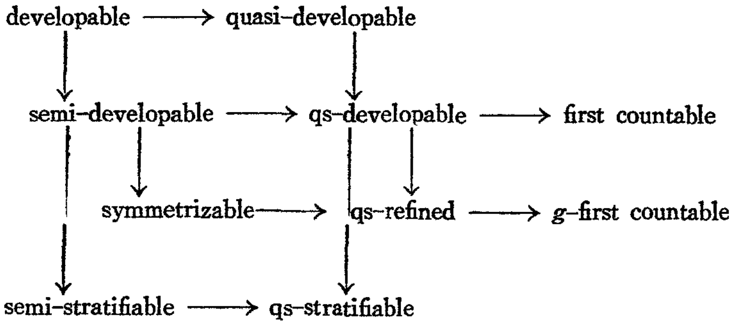
From this lemma we know that if a space is symmetrizable, it has a qs-refined sequence.

EXAMPLE 2. It is well known [3] that there is a symmetrizable space which is not first countable. Such a space has a qs-refined sequence but it is not qs-developable.

EXAMPLE 3. Let $\boldsymbol{Z}$ be the set of integers, and $\beta(\boldsymbol{Z})$ be the Stone-Čech compactification of Z and let $p \in \beta(Z)-Z$. The space $Z \cup \{p\}$ is a countable non-metrizable semi-stratifiable space [4]. It can be shown that this space is not g-first countable [9]. This means that a qs-stratifiable space [8] and even a semi-stratifiable space does not necessarily have a qs-refined sequence.

EXAMPLE 4. Let X be a qs-developable space. It is first countable and has a qs-refined sequence. We know that a symmetrizable space is semi-developable if it is first countable. But there is a qs-developable space which is not semi-developable (Example 3.1 of [7]). Therefore a space with a qs-refined sequence is not necessarily symmetrizable.

The relationships between some of the classes of spaces in [8] and the spaces which have qs-refined sequences are summerized in the following diagram.



A qs-developable space is semi-developable if and only if it is perfect [7]. A similar property holds between symmetrizable spaces and spaces with qs-refined sequences as the following theorem shows:

THEOREM 4. *A space with a qs-refined sequence is symmetrizable if it is perfect.*

Proof. Let $(\gamma_1, \gamma_2, \cdots\cdots)$ be a qs-refined sequence for X. For any $x \in \gamma_n^*$, $\mathrm{st}(x, \gamma_n) \subset \gamma_n^*$. Therefore γ_n^* is open for any $n \in Z^+$. Since X is perfect, there exist open sets U_{ni} such that $X-\gamma_n^* = \bigcap_{i=1}^{\infty} U_{ni}$. For each n and i, let $\gamma_{ni} = \gamma_n \cup \{U_{ni}\}$. Suppose M is an open set of X and $x \in M$. Then $x \in \mathrm{st}(x, \gamma_j) \subset M$ for some j. There exists some i such that $x \in U_{ji}$. For these i and j, $\mathrm{st}(x, \gamma_{ji}) \subset M$. Conversely let M be a subset of X such that for any $x \in M$, there exists n and i such that $x \in \mathrm{st}(x, \gamma_{ni}) \subset M$. Then either $x \in \mathrm{st}(x, \gamma_n) \subset M$ or $x \in U_{ni} \subset M$. Therefore M is open. This shows that the sequence $\{\gamma_{ni}: n, i \in Z^+\}$ is a qs-refined sequence for X. These facts and the Lemma 3 shows that X is symmetrizable.

Note that the converse of the Theorem 4 is not true. In fact there exists a symmetrizable space which is not perfect [2].

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