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WHAT IS THE FINSLER GEOMETRY ?

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It is the greatest honor and pleasure for me I could give a lecture at the opportunity of the 30th anniversary of the respective Korean Mathematical Society. I am studying Finsler geometry for a long time. It is quite sorry for me we have only one book of Finsler geometry written by H. Rund and published in 1959 from Springer Verlag. Thus Finsler geometry is not popular, but recently physicists and engineers have payed attention to Finsler geometry; I hear some problems of electric motors of "Sinkansen" are closely related to our geometry. We mathematicians have to promote the study of Finsler geometry for applications to such sciences.

Now the Finsler geometry is a kind of differential geometry and originated by P. Finsler (1894–1970) in 1918. It is usually considered as a generalization of the Riemannian geometry. In fact, B. Riemann, in his epoc-making lecture in 1854, already suggested a possibility of studying more general geometry than Riemannian geometry, but he said the geometrical meanings of quantities appearing in such a generalized space will not be clear and it can not produce any contribution to the geometry. Consequently all people had neglected for about 60 years to study such a geometry.

It seems Finsler didn't know Riemann's idea; he started the study of such a geometry from the standpoint of a geometrization of the variation calculus, his teacher Carathéodory's idea. In viewpoint of this idea we have some pioneers, for instance, W. Blaschke, G. M. Bliss, G. Landsberg, A. L. Underhill and etc. Finsler's famous thesis submitted to Göttingen University is really epoc-making and full of geometrical sense, but his theory was of old style and gave a generalization of the theory of curves and surfaces in an ordinary space.

The first period of the history of Finsler geometry began in 1924; three geometricians were almost simultaneously concerned with such a generalized space. J. H. Taylor and J. L. Synge introduced a special parallelism. but L.

Berwald (1883–1942) was the real originator of Finsler geometry. In 1928 Taylor gave the name "Finsler space" to the space with such generalized metric.

A Finsler space F^n of dimension n is a differentiable manifold such that the length s of a curve $x^i(t)$ of F^n is defined by the integral

$$s=\int L(x,\ dx/dt)\,dt.$$

The so-called fundamental function $L(x,y)$ $(=L(x^1,\cdots\cdots,x^n,\ y^1,\cdots\cdots,y^n))$ is supposed to be differentiable for $y\neq(0)$ and to satisfy the usual regularity conditions in the variation calculus:

(i) positively homogeneous: $L(x,py)=pL(x,y)$, $p>0$,

(ii) positive: $L(x,y)>0$, $y\neq(0)$,

(iii) $g_{ij}=(\partial^2L^2(x,y)/\partial y^i\,\partial y^j)/2$ is positive-definite.

Riemann himself further supposed the symmetry condition:

(iv) $L(x,-y)=L(x,y)$.

In case of Riemannian space the fundamental function $L(x,y)$ is given by

$$L(x,y)=\sqrt{g_{ij}(x)y^iy^j},$$

so that (i) and (iv) hold good. Further $g_{ij}(x)$ are nothing but the ones in (iii). Therefore $g_{ij}(x,y)$ in (iii) is called the fundamental tensor of the Finsler space F^n. It is well-known (iii) is too restrictive in application to physics in case of Riemannian geometry, so that (iii) as well as (ii) are usually not supposed. On the other hand, (iv) means the independence of the length of curves on their orientation and is neglected in Finsler geometry. As to this condition, Finsler sent me an interesting letter to me; in this letter he wrote "If we are concerned with the time-measure such as the one seen on the road sign, we can notice the length of the same way on the slope of a mountain is different, according to climbing or going down."

It is obvious that the quantities

$$C_{ijk}(x,y)=(\partial g_{ij}(x,y)/\partial y^k)/2$$

vanish, iff the space is Riemannian. In Finsler spaces the components of tensor fields, such as $g_{ij}(x,y)$ and $C_{ijk}(x,y)$, are functions of position x^i as well as the direction y^i. From this circumstances it follows that the partial derivatives $\partial S(x,y)/\partial x^i$ of a scalar field $S(x,y)$ are not regarded as components of a covariant vector. If we have a non-linear connection $(N^i{}_j(x,y))$, we can obtain the covariant vector field of the components

$$S|_i=\delta S/\delta x^i,\quad \delta/\delta x^i=\partial/\partial x^i-N^j{}_i\ \partial/\partial y^j.$$

Further, if we have quantities $F_j{}^i{}_k(x,,y)$ which obey the transformation rule similar to Christoffel symbols, the covariant derivatives $K^i{}_{j|k}$ of a Finslerian tensor field of (1, 1)-type are defined by

$$K^i{}_{j|k}=\delta K^i{}_j/\delta x^k+K^r{}_jF_r{}^i{}_k-K^i{}_rF_j{}^r{}_k.$$

On the other hand, it has been well known that the partial derivatives of components of a tensor field $K^i{}_j$ with respect to y^k yield a new tensor field, but we shall modify them as

$$K^i{}_j|_k=\partial K^i{}_j/\partial y^k+K^r{}_jC_r{}^i{}_k-K^i{}_rC_j{}^r{}_k,$$

where $C_j{}^i{}_k(x,y)$ are components of a tensor field of (1, 2)-type. The collection $(N^i{}_j,\ F_j{}^i{}_k,\ C_j{}^i{}_k)$ constitute a Finsler connection.

L. Berwald defined a Finsler connection from the fundamental function in viewpoint of the so-called geometry of paths. His method is very simple. but his connection is not metrical; the covariant derivarives $g_{ij|k}$ and $g_{ij}|_k$ of g_{ij} don't vanish.

The second period of the history of our geometry began in 1934 by publishing of the famous monograph of E. Cartan. He introduced a system of axioms to give uniquely a Finsler connection from the fundamental function $L(x,y)$. It seems that his axioms are rather artificial and introduced after foreseeing the desirable results. But, according to the recent study by M. Matsumoto in 1966, Cartan's axioms are equivalent to the following natural and elegant ones:

(i) $g_{ij|k}=0,\ \ g_{ij}|_k=0,$
(ii) $F_j{}^i{}_k=F_k{}^i{}_j,\ \ C_j{}^i{}_k=C_k{}^i{}_j,$
(iii) $N^i{}_j=y^kF_k{}^i{}_j.$

Cartan's $F_j{}^i{}_k$ are of complicated form; but $C_j{}^i{}_k$ are equal to $C_{jhk}g^{hi}$, where C_{jhk} are given as above. Cartan showed

$$G_j{}^i{}_k\ (=\text{Berwald's } F_j{}^i{}_k)=\Gamma^*{}_j{}^i{}_k\ (=\text{Cartan's } F_j{}^i{}_k)+C_j{}^i{}_{k|h}y^h.$$

In Cartan's theory of Finsler spaces we have three curvature tensors R_{hijk}, P_{hijk}, S_{hijk} and three torsion tensors $R^i{}_{jk}$ $(=y^hR_h{}^i{}_{jk})$, $P^i{}_{jk}$ $(=y^hP_h{}^i{}_{jk})$, $C_j{}^i{}_k$.

Since 1934 many mathematicians had been studying Finsler geometry along Cartan's line and we had various interesting results by authors of many countries, for instance, E. T. Davis (England), S. Golab (Poland), M. Haimovici (Roumania), H. Hombu (Japan), O. Varga (Hungary), V. V. Wagner (U. S. S. R.) and so on. On the other hand, G. Randers (U. S. A) introduced a special Finsler space to treat the so-called unified field theory of gravitation and electromagnetism, and many physicists had payed attention to Finsler geometry.

But the progress of studying of our geometry had losted the spead and finally arrived at the end, when Berwald died at a Jew Camp of Lodz in Poland in 1942. One of his posthumous papers was the greatest paper in our geometry.

The third period of the history of Finsler geometry began suddenly in 1951 from South Africa by young German H. Rund. He introduced a new parallelism from the standpoint of the so-called Minkowski geometry; Cartan did the parallelism from the standpoint of euclidean geometry. But the reviewers of Rund's paper, E. T. Davis (Math. Rev.) and A. Deicke (Zentralblatt.) indicated unfortunately Rund's parallelism was the same with Cartan's. By the stimulus of Rund, however, young German W. Barthel, A. Deicke, D. Laugwitz and R. Sulanke have very actively studied again Finsler geometry. Among various important results we are interested especially in

THEOREM (A. Deicke). *If* $C_i=g^{jk}C_{jki}=\partial(\log\sqrt{g})/\partial y^i=0$, *where* $g=\det(g_{ij})$, *then the space is Riemannian, provided that the condition* (ii) *on* L *be satisfied.*

Further, after L. Auslander, S. Kashiwabara, T. Ōtsuki and etc, H. Akbar-Zadeh developed a modern theory of Finsler spaces based on the theory of connections in fibre bundles in 1963 and M. Matsumoto completed the theory in 1970. On the other hand, various new ideas were given by many authors. In particular the study of the indicatrix $L(x,y)=1$ has been progressed. Among recent essential results the followings will be valuable:

THEOREM (F. Brickell in 1967). *If the curvature tensor* S_{hijk} *vanishes in*

F^n $(n>2)$, then F^n is Riemannian, provided that L be defined for $y \neq (0)$ *and satisfy the symmetry* (iv).

This theorem is never trivial, because $S_{hijk}=0$ in any two-dimensional Finsler space.

THEOREM (H. Akbar-Zadeh in 1963). *If the curvature tensor R_{hijk} is of the form $R_{hijk}=R(g_{hj}g_{ij}-g_{hk}g_{ij})$, $R \neq 0$, then the curvature tensor P_{hijk} is symmetric in j, k and $S_{hijk}=0$.*

In case of $R_{hijk}=0$ we have no essential result, except a theorem shown by M. Matsumoto.

THEOREM (M. Matsumoto in 1971 and 1975). (1) *The curvature tensor S_{hijk} of any three-dimensional Finsler space is witten in the form $S_{hijk}=S(h_{hj}h_{ik}-h_{hk}h_{ij})$, where $h_{ij}=L(\partial^2 L/\partial y^i \partial y^j)$ is the angular metric tensor.* (2) *S_{hijk} of any four-dimensional Finsler space is written in the form $S_{hijk}=h_{hj}M_{ik}+h_{ik}M_{hj}-h_{hk}M_{ij}-h_{ij}M_{hk}$, where M_{ij} is a symmetric tensor.* (3) *If the Ricci tensor $S_h{}^i{}_{ji}=0$ in a four-dimensional Finsler space, then $S_{hijk}=0$.*

In particular, (3) of the above theorem is important in relation to physics. We shall report here my friend Prof. Dr. Y. Takano is studying the field theory of elementary particles based on Finsler geometry.

Since 1970 the symposium of Finsler geometry has been held by the promotion of prof. Dr. A. Kawaguchi and myself every year, and now Japan may be said to be a centre of studying Finsler geometry in the world. I hope we can have good friends studying Finsler geometry in the nearest country from Japan.

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