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# MATHEMATICS ON TWO NORMED SPACES

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About 50 years ago, K. Menger [27] introduced the notion called a generalized metric. But many mathematicians had not paid attentions to Menger theory about generalized metrics. Very few mathematicians, for example, A. Wald, L. M. Blumenthal, W. A. Wilson, O. Haupt, C. Pauc, etc have developed Menger's idea. First these researches had a very close connection with the direct method of variation calculus (for details, see L. M. Blumenthal [2], [3] and C. Pauc [30]).

On the other hand, B. Vulich [35] introduced a notion of higher dimensional norm in linear spaces (1938). Unfortunately, this study had been neglected by many analysists for a long time.

A Froda's work [17] appeared in 1958. But a new development began with 1962. This contribution is due to S. Gahler [18]. This work was done under the guidance of W. Rinow. As well-known, in the present mathematics, one of the most important notions is the notion of metrics, which is fundamental in geometry and analysis and others. We certainly admit the importance of the notion of metrics.

However, we must recognize that the notion of metrics has a limitation. To pass the limitation, we need a new notion. One of the treatments is to consider a 2-metric space introduced by S. Gahler which is based on the researches of K. Menger.

The notion of a metric is to be regarded as a generalization of the notion of the distance between two points. A metric space is given by the following consideration. Let $X$ be a set with two variable function $d(x, y)$. If $d$ satisfies

(1) $d(x,y)=0$, if and olny if $x=y$, and $d(x,y)\geqslant 0$,

(2) $d(x,y)=d(y,x)$,

(3) $d(x,z)\leqslant d(x,y)+d(y,z)$,

then $(X,d)$ is called a *metric space*.

On the other hand, the notion of 2-metric spaces is obtained by a generalization of the notion of area. The area in the Euclidean plane is uniquely determined by given three points in the plane. Therefore each 2-simplex has its area. This idea is easily generalized to more higher dimensional figures, for example, a 3-simplex determined by four points in the Euclidean space. Then each simplex has a non-negative real number called a *volume*.

*The first problem is this: How to formulate a 2-metric space which is a generalization of the notion of area.*

Next, we must develop mathematics on 2-metric space. Can we obtain many important and interesting results about 2-metric spaces?

Another important problem is to formulate notion of differentiable manifolds based on 2-normed spaces to solve various variational problems. For the usual Banach space case, we have a good theory by S. Palais and others.

Unfortunately, the level of mathematics on 2-metric (or 2-normed) spaces is not so high, and the theory has not yet been developed until now. However, I think that this is a promising young branch in mathematics.

Let $X$ be an abstract set. We consider a mapping which is defined on the set of all triples of points $(x, y, z)$ of $X$ into the *reals* that satisfies

(1) there are three points $a, b, c$ such that
$\rho(a, b, c) \neq 0$,

(2) $\rho(x, y, z)=0$ if and only if at least two points of three points are equal.

(3) $\rho(x, y, z)=\rho(x, z, x)=\rho(y, z, x)\cdots\cdots$,

(4) $\rho(x, y, z)\leqslant\rho(x, y, u)+\rho(x, u, z)+\rho(u, y, z)$.

It is easily seen that $\rho$ *is non-negative.*. (3) means that $\rho$ *is symmetric about three variables* $x$, $y$, $z$. $\rho$ is called a *2-metric* on $X$.

Every Euclidean space of finite dimension$\geqslant 2$ has a 2-metric defined by

$$\rho(x, y, z)=\frac{1}{2}\sum_{i<j}\left\{\left|\begin{matrix} x_i & x_j & 1 \\ y_i & y_j & 1 \\ z_i & z_j & 1 \end{matrix}\right|^2\right\}^{\frac{1}{2}}$$

where $x_i$, $y_i$, $z_i$ are the coordinates of $x, y, z$ respectively.

This notion is easily generalized to the case of $n$-metrics, but we only consider the case of $n=2$, i.e., 2-metric case.

P. Cassens, B. A. Cassens and R. W. Freese [4] constructed the Euclidean geometry based on a 2-metric space. Then they considered a 2-metric space with the following condition:

(4) For $x, y, z, u$, if $\rho(x, y, z)=\rho(x, y, u)$, then $x=y$ or $\rho(y, z, u)=\rho(x, z, u)$.

For example, the set $\{z; \rho(x, y, z)=0\}$ is the *line passing through* $x$ *and* $y$ $(x \neq y)$.

For any 2-metric space, we can introduce a topology as follows: For each positive real $\varepsilon$ we define the $\varepsilon$-nbd(neighborhood) for two points $a$ and $b$ in $X$ as the set $U_\varepsilon(a, b)$ of all points $x$ in $X$ such that $\rho(a, b, x)<\varepsilon$. *Let* $V$ *be the set of all intersections* $\cap U_{\varepsilon_i}(a_i, b_i)$ *of finitely many* $\varepsilon_i$*-nbds of arbitrary points* $a_i$, $b_i$ *in* $X$. $\{V\}$ *forms a basis for the* 2*-metric topology of* $X$. This topology is called the *natural topology* or the *topology generated by the* 2*-metric* $\rho$ *in* $X$.

The totality of all set $W_\Sigma(a)$ defined by

$$W_\Sigma(a)=\cap U_{\varepsilon_i}(a, b_i)$$

with arbitrary $n$ and arbitrary pairs $\Sigma=\{(b_1, \varepsilon_1), (b_2, \varepsilon_2), \cdots\cdots, (b_n, \varepsilon_n)\}$ forms a complete system of nbds of the point $a$.

A point $a$ in a 2-metric space $X$ is called a *limit point of a set* $M$ *of* $X$, if for any $\Sigma=\{(b_1, \varepsilon_1), (b_2, \varepsilon_2), \cdots\cdots, (b_n, \varepsilon_n)\}$, there is a point $a_\Sigma$ in $M$, distinct from $a$, such that $a_\Sigma \in W_\Sigma(a)$.

$\rho(x, y, z)$ is continuous as a function of three points if and only if the space $X$ has the following property (this property is called *Property S*): For any two points $a$ and $b$ in $X$ and for a given positive $\varepsilon$ there are two nbds $U_a$ and $U_b$ of $a, b$ such that for any points $a'$ in $U_a$ and $b'$ in $U_b$, $\rho(a, b', a')<\varepsilon$.

S. Gähler [18] has considered a special 2-metric space having the following property: If for a sequence $a, a_1, a_2, \cdots$ of points in $X$, there exist two points $b$ and $c$ with $\rho(a, b, c) \neq 0$, $\lim_i \rho(a, b, a_i)=0$, $\lim_i \rho(a, c, a_i)=0$, then for each point $a'$ in $X$,

$$\lim_i \rho(a, a', a_i)=0.$$

This property is characterized by the following condition: Let $a, b, c$ be any points in $X$ such that $\rho(a, b, c) \neq 0$. Then $a$ is a limit point of a set $M$ in $X$, if there is a sequence of points $x_1, x_2, \cdots\cdots, x_n, \cdots$ in $M$ with

$$\lim_i \rho(a, b, x_i)=0, \qquad \lim_i \rho(a, c, x_i)=0.$$

Next we introduce a linear 2-normed space over reals.

By a *linear 2-normed space over the reals,* we mean a linear space $X$ in which to each pair of points $x$ and $y$ there exists a real number $\|x, y\|$ having the following properties:

(1) $\|x, y\|=0$ if and only if $x$ and $y$ are linearly dependent,

(2) $\|x, y\|=\|y, x\|$,

(3) For arbitrary real number $\alpha$, $\|\alpha x, y\|=|\alpha|\ \|x, y\|$,

(4) $\|x, y+z\|\leqslant\|x, y\|+\|x, z\|$.

For any 2-normed space $X$, we put $\rho(x, y, z)=\|y-x,\ z-x\|$. Then the 2-normed space $X$ becomes a 2-metric space.

The notions of convergences are introduced by A. White [36].

(1) *Cauchy sequence*: A sequence $\{x_n\}$ in a linear 2-normed space $X$ is called a *Cauchy sequence,* if there are $y$ and $z$ in $X$ such that $y$ and $z$ are linearly indepedent,

$$\lim_{m,n}\|x_m-x_n,\ y\|=0, \text{ and } \lim_{m,n}\|x_m-x_n,\ z\|=0.$$

(2) *Convergence sequence*: A sequence $\{x_n\}$ in a linear 2-normed space $X$ is called a *convergent sequence,* if there is an $x$ in $X$ such that

$$\lim_{m}\|x_m-x, y\|=0$$

for every $y$ in $X$.

(3) *2-Banach space*: A linear 2-normed space in which every Cauchy sequence is convergent is called a *2-Banach space.*

There are some interesting examples of 2-normed spaces.

EXAMPLE 1. Let $P_n$ denote the set of all real polynomials of degree $\leqslant n$ on the interval $[0, 1]$. We define addition and scalar multiplication in the usual way. Then $P_n$ is a linear space over reals. Let $\{x_n\}$ $(n=1, 2, \cdots, 2n+1)$ be given $2n+1$ points in $[0, 1]$. For $f$, $g$, we put

$$\|f,\ g\|=\sum_i|f(x_i)\times g(x_i)|$$

if $f, g$ are linearly independent, and $\|f, g\|=0$, if $f, g$ are linearly dependent. Then $P_n$ is a 2-Banach space.

On the other hand, there is a linear 2-normed space of dimension 3 which is not a 2-Banach space (such an example is given by A. White [35]). But every 2-normed space of dimension 2 is a Banach space when the underlying field is complete.

Next we shall explain a very important result about a 2-normed space.

Let $X$ be a 2-normed space, and let $a$ be a given non-zero element of $X$. We denote the 1-dimensional linear space generated by $a$ by $L(a)$. Then we can consider the quotient space $X/L(a)$. As well-known, this space $X/L(a)$ is also a linear space: For $x$ in $X$, let $x_a$ denote the equivalence class of $x$. Then the addition and the scalar multiplication are given by

$$x_a+y_a=(x+y)_a, \qquad \alpha x_a=(\alpha x)_a.$$

If $x_a=y_a$, then we have

$$|\|x,\ a\|-\|y,\ a\||\leqslant\|x-y,\ a\|=0.$$

Hence, $\|x,\ a\|=\|y,\ a\|\cdots$. Therefore the real valued function $\|\cdot\|_a$ given by $\|x_a\|_a=\|x,\ a\|$ is well-defined.

Then this new function is a norm on $X/L(a)$.

(1) $\|x_a\|_a=0$ if and only if $\|x, a\|=0$, if and only if $x\in L(a)$, if and only if $x_a=0$.

(2) $\|\alpha x_a\|_a=\|(\alpha x)_a\|_a=\|\alpha x, a\|=|\alpha|\ \|x, a\|=|\alpha|\ \|x_a\|_a$

(3) $\|x_a+y_a\|_a=\|(x+y)_a\|_a=\|x+y,\ a\|\leqslant\|x, a\|+\|y, a\|=\|x_a\|_a+\|y_a\|_a$

Hence $X/L(a)$ is a normed space.

THEOREM 1. *Let $X$ be a 2-normed space. For a non-zero element $a$ in $X$, the quotient space $X/L(a)$ is a normed space, where $L(a)$ is the 1-dimensional linear space generated by $a$.*

In general, this situation is formulated as follows;

From a new mathematical object and an equivalence relation, we have:

new mathematical object/equivalence relation

is isomorphic to a classical mathematical object.

In this case, two mathematical objects are essentially different. Then we say that *the new object is a non-Greek object of the classical object.* This idea was presented in my lecture [25] of the 7th conference of the Iranian Mathematical Society, Tabriz, March, 1976.

We have some useful and important cases of non-Greek objects. For example, consider the set of nonstandard finite real numbers, Then the set of all infinitesimals makes an ideal, and the quotient ring is the set of all real numbers. Therefore nonstandard finite reals is a non-Greek object of reals.

Another example comes from a Hjelmslev plane. A projective Hjelmslev-plane is a non-Greek object of an ordinary projective plane.

*It is very important to look at the classical mathematics from the new non-Greek object and to lead classical results from the new objects.*

For a 2-normed space, by Theorem 1, we can associate a normed space. An important problem is to discuss relationships between a 2-normed space and the associated normed space.

Among them, I will take up the notion of strictly convex 2-normed space.

DEFINITION 1. A 2-normed space $X$ is called *strictly convex*, if

$$\|x+y, z\| = \|x, z\| + \|y, z\|$$

and $z \in L(x, y)$ imply $y = ax$ for some $a > 0$, where $L(x, y)$ is the linear space generated by $x$, $y$.

We have the following characterizations of a strictly convex linear 2-normed space:

THEOREM 2. *For a linear 2-normed space, the following are equivalent*:

1. *The space is strictly convex,*

2. *For every non-zero element $a$, the associated normed space is strictly convex in the usual sense.*

3. $\|x+y, z\| = \|x, z\| + \|y, z\|$, $\|x, z\| = \|y, z\| = 1$, *and* $z \in L(x, y)$ *imply* $x = y$.

4. $\|x, z\| = \|y, z\| = 1$, $x = y$ *and* $z \in L(x, y)$ *imply* $\|(x+y)/2,\ z\| \leqslant 1$.

5. *If $F$ is a non-zero bounded bilnear form on $X$ $L(a)$ $(a \neq 0)$, $\|x, a\| = \|y, a\| = 1$ and $F(x, a) = F(y, a) = \|F\|$, then $x = y$, or $\|x, y\| = 0$, and $c = \pm(x-y)/\|x, y\|$.*

Let $Y, Z$ be linear subspaces of $X$, a bilnear form $F$ on $Y \times Z$ is called to be *bounded*, if there is a $K > 0$ such that

$$F(x, y) \leqslant K\|x, y\|$$

for every $(x, y) \in Y \times Z$, The infimum of such $K$ is denoted by $\|F\|$.

In my Note [23], I proved a fixed point theorem on mapping on a 2-normed space.

Let $X$ be a strictly convex linear 2-normed space of dimension greater than 2, $K$ a convex subset of $X$ and $T: K \to K$ non-expansive. i. e.,

$$\|T(x) - T(y),\ z\| \leqslant \|x - y, z\|$$

for all $z \in X$. Then the set of fixed points of $X$ is a convex set.

This result is generalized by C. Diminnie and A. White as follows: (see C. Diminnie and A. White [10])

THEOREM 3. *Let $K$ be a convex set which contains at least 2 elements and is not a subset of a line. Then, $T$ is non-expansive if and only if there is a $c \in K$ and there is a point $z_0 \in X$ such that $|c| \leqslant 1$ and $T(x)=cx+z_0$ for every $x \in K$.*

As Corollary of their result, my theorem is easily obtained.

THEOREM 4. *If $K$ is a convex subset of a 2-normed space and if $T:K \to X$ is non-expansive, then the set $F(T)$ of fixed points of $T$ is convex.*

There is an important class of mappings on a Banach space called a pseudo-contraction. This class has been studied by W. A. Kirk and his colleagues. We shall define a pseudo-contraction mapping on a linear 2-normed space.

Let $X$ be a 2-normed space and $D \subset X$. A mapping $U:D \to X$ is called to be *pseudo-contraction*, if for all $x$, $y \in D$ and all 0,

$$\|x-y, z\| \leqslant \|(1+r)(x-y)-r(U(x)-U(y)), z\|$$

for all $z \in D$.

Problem. Can we determine the form of $U$? Does $U:D \to D$ have a fixed point?

EXAMPLE. Let $X$ be a normed space with basis $\{e_n\}$ $(n=1, 2, \cdots\cdots)$. We define

$$\|e_m, e_n\| = \|e_n, e_m\| = \begin{cases} 0, & m=n, \\ 1, & \text{either } m=1, \text{ or } n=1, \\ \dfrac{1}{mn}, & \text{otherwise.} \end{cases}$$

For $x=\Sigma\alpha_n e_n$, $y=\Sigma\beta_m e_m$ (the summations are finite), we define

$$\|x, y\| = \frac{1}{2} \sum_{m,n} \text{abs} \begin{vmatrix} \alpha_m & \alpha_n \\ \beta_m & \beta_n \end{vmatrix} \|e_m, e_n\|.$$

Then $\|x, y\|$ is a 2-normed on $X$. However $\|e_2, e_1\| = \|e_3, e_1\| = 1$, $\|e_2+e_3, e_1\| = \|e_2, e_1\| + \|e_3, e_1\|$, and $e_1$ $L(e_2, e_3)$, but $e_2=e_3$, Hence $X$ is not strictly convex with respect to $\| \ \|$.

In this lecture, I only concerned with the real valued 2-norm. But we can consider a non-Archimedean valued 2-metric or 2-normed spaces. Then, for example, can we make the semi-Euclidean, the half-elliptic and the non-Legendre geometries? I think that this is an interesting problem.

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