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A NOTE ON PSEUDO-CONTRACTIVE MAPPINGS

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1. Introduction.

In this paper it is shown that if X is a convex Banach space, K is a compact convex subset of X, and U is a lipschitzian pseudo-contractive mapping of K into X such that $U(x)\in K$ when $x\in\partial K$. Then U has a fixed point in K.

DEFINITION 1.1. Let C be a non-empty subset of a Banach space$(X, \|\ \|)$. A mapping $T: C\to C$ is said to be *lipschitzian* if there is a constant $K>0$ such that $\|T(x)-T(y)\|\leqq K\|x-y\|$ for all x and y in C.

DEFINITION 1.2. Let C be a non-empty subset of a Banach space $(X, \|\ \|)$. A mapping $T: C\to C$ is said to be *non-expansive* if there is a constant $K=1$ such that $\|T(x)-T(y)\|\leqq K\|x-y\|$ for all x and y in C.

DEFINITION 1.3. Let C be a non-empty subset of a Banach space X. A mapping $T: C\to C$ is said to be *contraction* if there is a constant $0<K<1$ such that $\|T(x)-T(y)\|\leqq K\|x-y\|$ for all x and y in C.

Clearly, non-expansive mappings contain all contraction mappings as a proper subclass, and they form a proper subclass of the collection of all continuous mappings.

DEFINITION 1.4. [4]. Let X be a Banach space, let D be a subset of X. A mapping $U: D\to X$ is said to be *pseudo-contractive* if for all $u, v\in D$ and all $r>0$, $\|u-v\|\leqq\|(1+r)(u-v)-r(U(u)-U(v))\|$.

This class of mappings is easily seen to be larger than the class of non-expansive mappings, throughout our discussion, X will denote a Banach space, and for $K\subset X$, we use ∂K to denote the boundary of K, and $\lambda K=\{\lambda y : y\in K\}$. $\delta(K)=\sup\{\|x-y\| : x, y\in K\}$ to denote the diameter of K.

2. Main results.

In [2] the following Theorem has been proved.

THEOREM 2.1. [2]. *Let X be a Banach space, H a closed convex subset*

of X, and K a closed subset of H. If $T: K \to H$ is a contraction mapping, and if $T(x) \in K$ when $x \in \partial K$, then T has a (unique) fixed point in K.
A more interesting consequence of this theorem arises from taking $H=X$:

COROLLARY 2.2. *Let K be a closed subset of convex Banach space X. If $T: K \to X$ is a contraction mapping, and if $T(x) \in K$ when $x \in \partial K$, then T has a (unique) fixed point in K.*

W. G. Dotson, JR, and W. R. Mann. have proved the following theorem.

THEOREM 2.3. [1]. *Let X be a Banach space, let C be compact convex subset of X. If $T: C \to C$ is a non-expansive mappping, then T has a fixed point in C.*

THEOREM 2.4. *Let X be a convex Banach space, let K be a compact convex subset of X. Let U be a lipschitzian pseudo-contractive mapping of K into X such that $U(x) \in K$ when $x \in \partial K$, then U has a fixed point in K.*

Proof. Since U is pseudo-contractive: For all u, v in K and all $r>0$, $\|u-v\| \leqq \|(1+r)(u-v) \ -r(U(u)-U(v))\|$—(1). Taking $\lambda=\frac{r}{1+r}$. (1) equivalent to $(1-\lambda)\|u-v\| \leqq \|(u-v)-\lambda(U(u)-U(v))\|$. Then $\|(I-\lambda U)(u)-(I-\lambda U)(v)\| \geqq (1-\lambda)\|u-v\|$, $\lambda>0$. Put $T_\lambda=I-\lambda U$. Then $\|T_\lambda(u)-T_\lambda(v)\| \geqq (1-\lambda)\|u-v\|$, for all u, v in K.

Since U is lipschitzian, there is $C>0$ such that $\|U(u)-U(v)\| \leqq C\|u-v\|$. Select $\lambda>0$ such that $\lambda C<1$ and $\lambda<1$, and let $U_\lambda=\lambda U$. Then $\|U_\lambda(u)-U_\lambda(v)\|=\lambda\|U(u)-U(v)\| \leqq \lambda C\|u-v\|$. Therefore U_λ is a contractive mapping on K. By $\|T_\lambda(u)-T_\lambda(v)\| \geqq (1-\lambda)\|u-v\|$, $u, v \in K$. Hence $(1-\lambda)T_\lambda^{-1}$ is a non-expansive on it's domain. Now let $y^* \in (1-\lambda)K=\{(1-\lambda)y : y \in K\}$ and consider $\bar{U}_\lambda : K \to X$ defined by $\bar{U}_\lambda(x)=U_\lambda(x)+y^*$, $x \in K$. For $x \in \partial K$, then $U(x) \in K$, Thus $\bar{U}_\lambda(x)=U_\lambda(x)+(1-\lambda)y'$ for some y' in K. Since $K=$ convex. Therefore $\bar{U}_\lambda(x)$ in K whenever $x \in \partial K$. Since $\|\bar{U}_\lambda(u)-\bar{U}_\lambda(v)\|=\|\lambda U(u)-\lambda U(v)\|$, we have $\bar{U}_\lambda$ is a contractive mapping. Thus $\bar{U}_\lambda$ is a contraction mapping satisfying the assumptions of Corollary 2, 2. Hence there exists x^* in K such that $\bar{U}_\lambda(x^*)=x^*$.

Hence $U_\lambda(x^*)+y^*=x^*$; that is $(I-U_\lambda)(x^*)=y^*$. Note that $\lambda U=I-T_\lambda$. Thus we have proved $T_\lambda(K) \supset (1-\lambda)K$; $T_\lambda^{-1}(1-\lambda)K \subset K$. Then $(1-\lambda)T_\lambda^{-1} : (1-\lambda)K \to (1-\lambda)K$ is a non-expansive. Since $(1-\lambda)K$ is closed and $(1-\lambda)K$ is compact. By Theorem 2.3. $(1-\lambda)T_\lambda^{-1}$ has a fixed point in $(1-\lambda)K$. Thus there exists $z \in (1-\lambda)K$ such that $(1-\lambda)T_\lambda^{-1}(z)=z$. Then $T_\lambda^{-1}(z)=\frac{z}{1-\lambda}=\frac{(1-\lambda)y}{1-\lambda}=y$, for some $y \in K$. Hence $(I-\lambda U)(y)=y-\lambda U(y)=z=(1-\lambda)y$, Therefore $y-\lambda U(y)=y-\lambda y$. Hence $U(y)=y$.

Clearly, We have the following Corollary.

COROLLARY 2.5. *Let X be a convex Banach space. Let K be non-empty compact convex subset of X, if U is a lipschitzian pseudo-contractive mapping of K into itself. Then U have a fixed point in K.*

LEMMA 2.6. *Let X be a Banach space, and let U be a continuous mapping of X into itself such that U satisfies the inequalities*

(1) $\|U(x)-U(y)\| \leq \alpha \max \{\|x-y\|, [\beta\|x-U(x)\|+\gamma\|y-U(y)\|], \frac{1}{2}[\|x-U(y)\|+\|y-U(x)\|]\}$, *for all x and y in X, $\beta+\gamma=1$ and for some $0<\alpha<1$.*

(2) $\operatorname{Inf}\{\|x-U(x)\| : x \in X\}=0$. *Then U has a (unique) fixed point.*

Proof. Consider the set C_m defined by $C_m=\left\{x \in X : \|x-U(x)\| \leq \frac{1}{m}\right\}$. From (2) and the continuity of U, we get C_m is closed and non-empty for each $m=1, 2, \cdots\cdots$. Now if $x, y \in C_m$, then $\|x-y\| \leq \|x-U(x)\|+\|U(x)-U(y)\|+\|y-U(y)\| \leqq \|x-U(x)\|+\|y-U(y)\|+\alpha \max \{\|x-y\|, [\beta\|x-U(x)\|+\gamma\|y-U(y)\|], \frac{1}{2}[\|x-U(y)\|+\|y-U(x)\|]\}$

$$\leqq \frac{2}{m}+\alpha[\frac{1}{2}(\|x-y\|+\|y-x\|+\|x-U(x)\|+\|y-U(y)\|)+\frac{1}{2}((2\beta-1)\|x-U(x)\|+(2\gamma-1)\|y-U(y)\|)]$$

$$\leqq \frac{2}{m}+\alpha\|x-y\|+\frac{\alpha}{m}+\frac{1}{2}\left\{(2\beta-1)\frac{1}{m}+(2\gamma-1)\frac{1}{m}\right\}$$

$$=\frac{2}{m}+\alpha\|x-y\|+\frac{\alpha}{m}.$$

Thus $(1-\alpha)\|x-y\| \leqq \frac{2+\alpha}{m}$. Therefore $\|x-y\| \leqq \frac{2+\alpha}{m(1-\alpha)}$.

Hence $\delta(C_m) \leqq \frac{(2+\alpha)}{m(1-\alpha)}$.

Thus the family of sets $\{C_m\}_{m=1}^{\infty}$ is nested family of closed sets for which $\delta(C_m) \to 0$ as $m \to \infty$.

The intersection of these sets contains a single point x_0. Since $U(C_m) \subseteq C_m$ for all m, and $U(x_0)=U(\bigcap_{m=1}^{\infty} C_m)=\bigcap_{m=1}^{\infty} U(C_m) \subseteq \bigcap_{m=1}^{\infty} C_m=\{x_0\}$.

Hence x_0 is a fixed point of U.

Next we prove the uniqueness of the fixed point. Let y_0 be another fixed point of U, i.e. $U(y_0)=y_0$, different from x_0. From the inequality

(1) of the Lemma 2.6.

We have $\|x_0-y_0\| \leqq \alpha \max\{\|x_0-y_0\|, [\beta\|x_0-U(x_0)\|+\|y_0-U(y_0)\|], \frac{1}{2}[\|x_0-U(y)\|+\|y_0-U(x_0)\|]\} \leqq \alpha\|x_0-y_0\| \leqq \|x_0-y_0\|$ which is a contradiction. Hence $x_0=y_0$.

REMARK: If we put $\beta=\frac{1}{2}$, $\gamma=\frac{1}{2}$, the theorem (1) in [3] follows from our Lemma 2.7.

REMARK: It may be noted that the Lemma 2.6 can be proved when U satisfies the condition (1) only.

The following Lemma 2.8 follows by theorem in [5].

LEMMA 2.7. *Let X be a Banach space. Let $U: X \to X$ a continuous pseudo-contractive mapping and suppose that for some $\delta>0$ the set $\{x \in X : \|x-U(x)\| \leqq \delta\}$ is non-empty and bounded. Then* $\operatorname{Inf}\{\|x-U(x)\| : x \in X\}=0$.

By Lemma 2.6 and Lemma 2.7 we have the following Theorem 2.8.

THEOREM 2,8. *Let X be a Banach space, let $U: X \to X$ a continuous pseudo-contractive mapping such that*

$$\|U(x)-U(y)\| \leqq \alpha \max\{\|x-y\|, [\beta\|x-U(x)\|+\gamma\|y-U(y)\|], \frac{1}{2}[\|x-U(y)\|+\|y-U(x)\|]\}$$

for all $x, y \in X$. $\beta+\gamma=1$ and for some $0<\alpha<1$ and suppose that for some $\delta>0$ the set $\{x \in X : \|x-U(x)\| \leqq \delta\}$ is nonempty and bounded. Then T has a unique fixed point.

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