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A STUDY ON THE OPERATOR L_K

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1. Introduction.

Naimark [3], Krall [2], and Kim [1] discussed an operator generated by a differential expression

$$ly=-y''+q(x)y,\ \ 0\leq x<\infty,\ \ \int_0^\infty |q(x)|dx<\infty$$

and various boundary conditions. In 1954, Naimark [3] discussed an operator L_θ generated by the differential expression $ly=-y''+q(x)y$, where $\int_0^\infty e^{\varepsilon x}|q(x)|dx<\infty$ for some $\varepsilon>0$, $0\leq x<\infty$ and the boundary condition $y'(0)-\theta y(0)=0$, where θ is a fixed number. In 1965, Krall [2] discussed the differential operator L generated by the differential expression $ly=-y''+q(x)y$, where $\int_0^\infty |q(x)|dx<\infty$, $0\leq x<\infty$ and the boundary condition $\int_0^\infty K(x)y(x)dx=\beta y(0)-\alpha y'(0)$, where $K(x)$ is in $L^2(0,\infty)$ and $|\alpha|^2+|\beta|^2\neq 0$. In 1973, Kim [1] discussed an operator L_K generated by a differential expression $ly=y''-h(x)(a_1y(0)+a_2y'(0))$, where $h(x)$ is in $L^2(0,\infty)$ and $|a_1|^2+|a_2|^2\neq 0$, and the boundary condition $\int_0^\infty K(x)y(x)dx=b_1y(0)+b_2y'(0)$, where $K(x)$ is in $L^2(0,\infty)$ and $|b_1|^2+|b_2|^2\neq 0$. In this paper, we want to discuss the operator L_K further. We shall discuss the expansion of the green's function $G(x,\xi,\lambda)$ of the operator $L_K+\lambda$ and the eigenfunction expansion of a certain function.

2. Expansion of the green's function $G(x,\xi,\lambda)$ of the operator $L_K+\lambda$.

We define the differential expression $ly=y''-h(x)(a_1y(0)+a_2y'(0))$, for all functions $y\in C^2[0,\infty)$, where $h(x)$ is an arbitrary measurable function in $L^2(0,\infty)$ and $|a_1|^2+|a_2|^2\neq 0$.

Let D be the set of those functions $f(x)$ on $[0,\infty)$ satisfying

(1) $f(x)$ is in $L^2(0,\infty)$

(2) $f'(x)$ exists and is absolutely continuous on every finite subinterval

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$[0, b]$ of $[0, \infty)$

(3) $lf(x)$ is in $L^2(0, \infty)$.

Let $K(x)$ be a function in $L^2(0, \infty)$ and let b_1 and b_2 be complex numbers such that $|b_1|^2+|b_2|^2\neq 0$. Let D_K be the set of those functions $f(x)$ satisfying

(1) $f(x)$ is in D

(2) $\int_0^\infty K(x)f(x)\,dx = b_1 f(0) - b_2 f'(0)$.

Now we define the operator L_K by $L_K f(x) = lf(x)$ for all functions $f(x)$ in D_K. We shall discuss the operator L_K in the following way;

(1) We find L^2 solution of $ly+\lambda y=0$

(2) We find a particular solution of $L_K y+\lambda y=f(x)$

(3) We find the green's function of the operator $L_K+\lambda$

(4) We expand the green's function which is obtained

(5) We expand a certain function using the eigenfunctions of the operator L_K.

THEOREM 1. *Linearly independent L^2 solution of $ly+\lambda y=0$ is given by*

$$(1)\quad y(x, s) = e^{isx} + \left[e^{-isx}\int_x^\infty \frac{e^{is\xi}}{2is}h(\xi)\,d\xi + e^{isx}\int_0^x \frac{e^{-is\xi}}{2is}h(\xi)\,d\xi\right]\left[\frac{a_1+isa_2}{1+is\alpha a_2-\alpha a_1}\right]$$

where $s=\sqrt{\lambda},\ 0\leq \arg s<\pi,\ s=\sigma+i\tau,\ \alpha=\int_0^\infty \frac{e^{isx}}{2is}h(x)\,dx.$

Proof. See Kim [1].

THEOREM 2. *For the eigenvalue problem $L_K y+\lambda y=0$, the eigenvalues are $\lambda = s^2$, where* $\mathrm{Im}\, s>0$ *and s is a solution of*

$$(2)\quad 2is\delta\zeta + \int_0^\infty K(x)v_1(x, s)\,dx\,(a_1+isa_2) - (b_1-isb_2) - 2is\alpha(a_1b_2+a_2b_1) = 0,$$

where

$$\delta = \int_0^\infty \frac{e^{isx}}{2is}K(x)\,dx,\quad \zeta = 1+is\alpha a_2 - \alpha a_1 \neq 0,$$

$$v_1(x, s) = e^{-isx}\int_x^\infty \frac{e^{is\xi}}{2is}h(\xi)\,d\xi + e^{isx}\int_0^x \frac{e^{-is\xi}}{2is}h(\xi)\,d\xi.$$

Proof. See Kim [1].

If we define β by

$$\beta=2is\delta+\frac{a_1+isa_2}{\zeta}\int_0^\infty K(x)v_1(x,\ s)dx-\frac{b_1-isb_2}{\zeta}-\frac{2is\alpha(a_1b_2+a_2b_1)}{\zeta}$$

we can write (2) as a form

$$\beta\zeta=0. \tag{3}$$

THEOREM 3. *If $L_Ky+\lambda y=0$ has only trivial solution, then for any function $f(x)$ in $L^2(0,\infty)$, there exists a solution of the equation $L_Ky+\lambda y=f(x)$ and the solution is expressed by*

$$y=\int_0^\infty G(x,\xi,\lambda)f(\xi)d\xi, \tag{4}$$

where

$$G(x,\xi,\lambda)=V_1(x,\xi,\lambda)+V_2(x,\xi,\lambda)$$

and

$$\begin{aligned}V_1(x,\xi,\lambda)=&\frac{\gamma}{2is}e^{isx}\ e^{is\xi}+\frac{\gamma}{2is}\ \frac{a_1+isa_2}{\zeta}v_1(x,s)e^{is\xi}\\&-\frac{1}{\beta}e^{isx}v_2(\xi,s)-\frac{a_1+isa_2}{\beta\zeta}v_1(x,s)v_2(\xi,s)\\&+\frac{1}{2is}\ \frac{b_1+isb_2}{\beta\xi}e^{isx}e^{is\xi}+\frac{1}{2is\beta}\ \frac{(b_1+isb_2)(a_1+isa_2)}{\zeta^2}\\&v_1(x,s)e^{is\xi}\\&+\frac{1}{2is}\ \frac{a_1-isa_2}{\zeta}v_1(x,s)e^{is\xi},\end{aligned}$$

$$V_2(x,\xi,\lambda)=\frac{1}{2is}e^{isx>}e^{-isx<},$$

and

$$v_2(x,s)=e^{-isx}\int_X^\infty\frac{e^{is\xi}}{2is}K(\xi)d\xi+e^{isx}\int_0^X\frac{e^{-is\xi}}{2is}K(\xi)d\xi$$

$$\gamma=-\frac{a_1-isa_2}{\beta\zeta}\int_0^\infty K(x)v_1(x,s)dx$$

$$e^{isx>}e^{-isx<}=\begin{cases}e^{isx}e^{-is\xi} & x>\xi\\ e^{is\xi}e^{-isx} & x<\xi\end{cases}$$

Proof. See Kim [1].

Now we consider the expansion of the green's function $G(x,\xi,\lambda)$. We assume that $h(x)$, $K(x)$ are in $L^2(0,\infty)$ and for simplicity, $\beta(s)$ is never zero for all real s and the complex zeros of $\beta(s)$ are simple zeros. We shall use the residue theorem to integrate about a closed contour which may contain the boundary of the analytic function being considered as long as the function is continuous on that part of the boundary.

We choose the contour C in the λ plane to be the course consisting of the large circle C_R, C_+, and C_- (see).

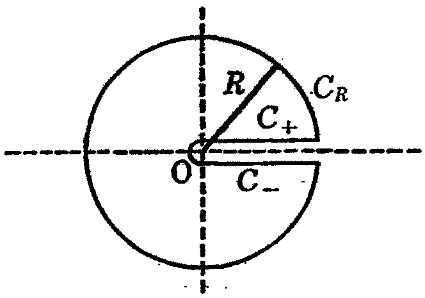

Figure 1

From the equation (2), we can obtain the following theorem easily.

THEOREM 4. *$\lambda=s^2$ is an eigenvalue of the operator L_K if and only if $\beta(\sqrt{\lambda})=0$.*

We first consider $V_1(x,\xi,\lambda)$. Let $\lambda_1=s_1^2$ be a simple zero of $\beta(\sqrt{\lambda})$.

Then λ_1 is a simple pole of the function $V_1(x,\xi,\lambda)$ and

$$V_1(x,\xi,\lambda)=\frac{R(x,\xi)}{\lambda-\lambda_1}+G_2(x,\xi,\lambda) \tag{5}$$

where $G_2(x,\xi,\lambda)$ is analytic in a neighborhood of the point $\lambda_1=s_1^2$. By the residue theorem,

$$R(x,\xi)=\frac{\varphi(x)}{\beta'(s_1)}, \tag{6}$$

where

$$\begin{aligned}(7)\quad \varphi(x)=&-\frac{a_1-is_1a_2}{2is_1(1+is_1\alpha a_2-\alpha a_1)}\int_0^\infty K(x)v_1(x,s_1)\,dx e^{is_1x}e^{is_1\xi}\\&+\frac{b_1+is_1b_2}{2is_1(1+is_1\alpha a_2-\alpha a_1)}e^{is_1x}e^{is_1\xi}+\frac{(a_1b_2+a_2b_1)+(a_1-is_1a_2)\delta}{1+is_1\alpha a_2-\alpha a_1}\\&v_1(x,s_1)e^{is_1\xi}\\&-\frac{a_1+is_1a_2}{1+is_1\alpha a_2-\alpha a_1}v_1(x,s_1)v_2(\xi,s_1)-e^{is_1x}v_2(\xi,s_1).\end{aligned}$$

For a fixed ξ, $\varphi(x)$ becomes an eigenfunction of the operator L_K corresponding to an eigenvalue λ_1. We need to prove that $\varphi(x)$ satisfies the differential equation $y''+\lambda_1 y=h(x)(a_1 y(0)+a_2 y'(0))$ and the boundary condition $\int_0^\infty K(x)y(x)dx=b_1 y(0)-b_2 y'(0)$. From the equation (7), we obtain the relation,

$$(8)\quad \varphi''+\lambda_1\varphi=\frac{(a_1b_2+a_2b_1)+(a_1-is_1a_2)\delta}{1+is_1\alpha a_2-\alpha a_1}h(x)e^{is_1\xi}-\frac{a_1+is_1a_2}{1+is_1\alpha a_2-\alpha a_1}h(x)v_2(\xi,s_1).$$

To compute $a_1\varphi(0)+a_2\varphi'(0)$, we use the expression (7) and the expression (2), and obtain

$$(9)\quad a_1\varphi(0)+a_2\varphi'(0)=\frac{(a_1b_2+a_2b_1)+(a_1-is_1a_2)\delta}{1+is_1\alpha a_2-\alpha a_1}e^{is_1\xi}-\frac{a_1+is_1a_2}{1+is_1\alpha a_2-\alpha a_1}v_2(\xi,s_1).$$

Therefore $\varphi(x)$ satisfies the differential equation

$$(10)\quad \varphi''+\lambda_1\varphi=h(x)(a_1\varphi(0)+a_2\varphi'(0)).$$

Now we wish to show that $\varphi(x)$ satisfies the above boundary condition. If we substitute $\varphi(x)$ in $\int_0^\infty K(x)\varphi(x)dx$, we have

$$(11)\quad \int_0^\infty K(x)\varphi(x)dx=\frac{b_1+is_1b_2}{1+is_1\alpha a_2-\alpha a_1}e^{is_1\xi}\delta+\frac{a_1b_2+a_2b_1}{1+is_1\alpha a_2-\alpha a_1}e^{is_1\xi}\int_0^\infty K(x)v_1(x,s_1)dx$$
$$-\frac{a_1+is_1a_2}{1+is_1\alpha a_2-\alpha a_1}v_2(\xi,s_1)\int_0^\infty K(x)v_1(x,s_1)dx-v_2(\xi,s_1)2is_1\delta.$$

Using the relation (2), if we substitute an equivalent form for $\int_0^\infty K(x)v_1(x,s_1)dx$, the equation (11) becomes

$$(12)\quad \int_0^\infty K(x)\varphi(x)dx=\frac{(a_1b_2+a_2b_1)\{(b_1-is_1b_2)+2is_1\alpha(a_1b_2+a_2b_1)\}}{(1+is_1\alpha a_2-\alpha a_1)(a_1+is_1a_2)}e^{is_1\xi}$$
$$+\frac{a_1b_1-s_1^2a_2b_2-is_1(a_1b_2+a_2b_1)(1-2\alpha a_1+2is_1\alpha a_2)}{(1+is_1\alpha a_2-\alpha a_1)(a_1+is_1a_2)}e^{is_1\xi}\delta$$
$$-\frac{(b_1-is_1b_2)+2is_1\alpha(a_1b_2+a_2b_1)}{1+is_1\alpha a_2-\alpha a_1}v_2(\xi,s_1).$$

Now consider $b_1\varphi(0)-b_2\varphi'(0)$. We compute

$$(13)\quad b_1\varphi(0)-b_2\varphi'(0)=-\frac{(a_1-is_1a_2)e^{is_1\xi}\int_0^\infty K(x)v_1(x,s_1)dx}{2is_1(1+is_1\alpha a_2-\alpha a_1)}(b_1-is_1b_2)$$
$$+\frac{b_1+is_1b_2}{2is_1(1+is_1\alpha a_2-\alpha a_1)}e^{is_1\xi}(b_1-is_1b_2)$$
$$+\frac{(a_1b_2+a_2b_1)+(a_1-is_1a_2)\delta}{1+is_1\alpha a_2-\alpha a_1}\alpha(b_1+is_1b_2)e^{is_1\xi}$$
$$-\frac{a_1+is_1a_2}{1+is_1\alpha a_2-\alpha a_1}\alpha v_2(\xi,s_1)(b_1+is_1b_2)-v_2(\xi,s_1)$$
$$(b_1-is_1b_2).$$

Using the equation (2), if we substitute an equivalent form for $\int_0^\infty K(x)$ $v_1(x,s_1)dx$, the equation (13) becomes

$$(14)\quad b_1\varphi(0)-b_2\varphi'(0)=\frac{(a_1b_2+a_2b_1)\{(b_1-is_1b_2)+2is_1\alpha(a_1b_2+a_2b_1)\}}{(1+is_1\alpha a_2-\alpha a_1)(a_1+is_1a_2)}e^{is_1\xi}$$
$$+\frac{a_1b_1-s_1^2a_2b_2-is_1(a_1b_2+a_2b_1)(1-2\alpha a_1+2is_1\alpha a_2)}{(1+is_1\alpha a_2-\alpha a_1)(a_1+is_1a_2)}e^{is_1\xi}\delta$$
$$-\frac{2is_1\alpha(a_1b_2+a_2b_1)+(b_1-is_1b_2)}{1+is_1\alpha a_2-\alpha a_1}v_2(\xi,s_1).$$

Comparing the equations (12) and (14), we see that $\varphi(x)$ satisfies the boundary condition

$$\int_0^\infty K(x)\varphi(x)dx=b_1\varphi(0)-b_2\varphi'(0).$$

Therefore the theorem is proved.

Now we go back to the equation (6). Since λ_1 is a simple zero of the function $\beta(\sqrt{\lambda})$, there is only one eigenfunction $y_1(x)$ corresponding to it, up to a factor independent of x, for the operator L_K. $R(x,\xi)=\varphi(x)/\beta'(\sqrt{\lambda_1})$ is also an eigenfunction corresponding to an eigenvalue $\sqrt{\lambda_1}=s_1$. Therefore

$$(15)\qquad R(x,\xi)=a(\xi)y_1(x).$$

We wish to determine the function $a(\xi)$. Let $G^*(x,\xi,\lambda)=\overline{G(\xi,x,\lambda)}$. Then $G^*(x,\xi,\lambda)$ becomes the green's function for the operator $L_K^*+\bar{\lambda}$, where L_K^* is the adjoint operator of the operator L_K. Since $G(x,\xi,\lambda)$ is expressed by

$$(16)\qquad G(x,\xi,\lambda)=\frac{R(x,\xi)}{\lambda-\lambda_1}+G_2(x,\xi,\lambda)+V_2(x,\xi,\lambda),$$

$G^*(x, \xi, \lambda)$ has the form

$$G^*(x, \xi, \lambda) = \frac{\overline{R(\xi, x)}}{\bar{\lambda} - \bar{\lambda}_1} + \overline{G_2(x, \xi, \lambda)} + \overline{V_2(x, \xi, \lambda)}. \tag{17}$$

So, for a fixed ξ, $\overline{R(\xi, x)}$ is an eigenfunction of the operator $L_K{}^*$, corresponding to the eigenfunction $\bar{\lambda}_1$. If we denote one of these functions by $z_1(x)$, we then have

$$\overline{R(\xi, x)} = b(\xi) z_1(x).$$

Hence

$$R(x, \xi) = \overline{b(x) z_1(\xi)}. \tag{18}$$

Comparing this with the equation (15), we find

$$R(x, \xi) = c y_1(x) \overline{z_1(\xi)}. \tag{19}$$

Now we try to determine the constant c.
For the equation

$$G(x, \xi, \lambda) = \frac{R(x, \xi)}{\lambda - \lambda_1} + G_2(x, \xi, \lambda) + V_2(x, \xi, \lambda), \tag{20}$$

multiply both sides by $(\lambda - \lambda_1)$. Then we have

$$\begin{aligned}(\lambda-\lambda_1) G(x, \xi, \lambda) &= R(x, \xi) + (\lambda-\lambda_1) G_2(x, \xi, \lambda) + (\lambda-\lambda_1) V_2(x, \xi, \lambda) \\ &= c y_1(x) \overline{z_1(\xi)} + (\lambda-\lambda_1) G_2(x, \xi, \lambda) + (\lambda-\lambda_1) V_2(x, \xi, \lambda).\end{aligned}$$

Multiplying both sides by $y_1(\xi)$ and integrating it, we have

$$\begin{aligned}(\lambda-\lambda_1) \int_0^\infty G(x, \xi, \lambda) y_1(\xi) d\xi = c y_1(x) \int_0^\infty \overline{z_1(\xi)} y_1(\xi) d\xi + (\lambda-\lambda_1) \int_0^\infty G_2(x, \xi, \lambda) \\ y_1(\xi) d\xi \\ + (\lambda-\lambda_1) \int_0^\infty V_2(x, \xi, \lambda) y_1(\xi) d\xi.\end{aligned} \tag{21}$$

Taking limit both sides, we have

$$\lim_{\lambda \to \lambda_1} (\lambda-\lambda_1) \int_0^\infty G(x, \xi, \lambda) y_1(\xi) d\xi = c y_1(x) \int_0^\infty y_1(\xi) \overline{z_1(\xi)} d\xi \tag{22}$$

On the other hand

$$(L_K + \lambda) y_1(x) = -\lambda_1 y_1(x) + \lambda y_1(x) = (\lambda - \lambda_1) y_1(x)$$

and

$$(L_K + \lambda)^{-1} y_1(x) = \int_0^\infty G(x, \xi, \lambda) \; y_1(\xi) d\xi = \frac{1}{\lambda - \lambda_1} y_1(x). \tag{23}$$

Substituting the equation (23) in (22), we get

$$y_1(x)=cy_1(x)\int_0^\infty y_1(\xi)\overline{z_1(\xi)}\,d\xi.$$

$$c=\frac{1}{\int_0^\infty y_1(\xi)\overline{z_1(\xi)}\,d\xi}$$

Thus $R(x,\xi)$ in (19) becomes

$$R(x,\xi)=\frac{y_1(x)\,z_1(\xi)}{\int_0^\infty y_1(\xi)\overline{z_1(\xi)}\,d\xi}. \tag{24}$$

Using this argument, we have the following theorems.

THEOREM 5. *For every simple zero* λ_1 *of the function* $\beta(\sqrt{\lambda})$,

$$G(x,\xi,\lambda)=\frac{y_1(x)\overline{z_1(\xi)}}{(\lambda-\lambda_1)\int_0^\infty y_1(\xi)\overline{z_1(\xi)}\,d\xi}+G_2(x,\xi,\lambda)+V_2(x,\xi,\lambda) \tag{25}$$

where $G_2(x,\xi,\lambda)$, $V_2(x,\xi,\lambda)$ *are analytic in a neighborhood of the point* λ_1.

THEOREM 6. *Let* $\{\lambda_k\}_{k=1}^n$ *be the set of simple zeros of* $\beta(\sqrt{\lambda})$, *i.e.* $\{\lambda_k\}_{k=1}^n$ *be the set of simple eigenvalues of the operator* L_K. *Then we have*

$$G(x,\xi,\lambda)=\sum_{k=1}^n\frac{y_k(x)\overline{z_k(\xi)}}{(\lambda-\lambda_k)\int_0^\infty y_k(\xi)\overline{z_k(\xi)}\,d\xi}+G_{n+1}(x,\xi,\lambda)+V_2(x,\xi,\lambda)$$

where $G_{n+1}(x,\xi,\lambda)$, $V_2(x,\xi,\lambda)$ *are analytic in a neighborhood of each of* $\{\lambda_k\}_{k=1}^n$, *and*

$$V_1(x,\xi,\lambda)=\sum_{k=1}^n\frac{y_k(x)\overline{z_k(\xi)}}{(\lambda-\lambda_k)\int_0^\infty y_k(\xi)\overline{z_k(\xi)}\,d\xi}+G_{n+1}(x,\xi,\lambda).$$

Now we integrate $\dfrac{V_1(x,\xi,\lambda)}{\lambda-\lambda_0}$ around the contour in Fig. 1, where $\lambda_0=s_0^2$ is in the interior of C_R and is not an eigenvalue of L_K. By the residue theorem, the contour integral of

$$\frac{V_1(x,\xi,\lambda)}{\lambda-\lambda_0}$$

is

$$\int_{C+}\frac{V_1(x,\xi,\lambda)}{\lambda-\lambda_0}d\lambda+\int_{C_R}\frac{V_1(x,\xi,\lambda)}{\lambda-\lambda_0}d\lambda+\int_{C-}\frac{V_1(x,\xi,\lambda)}{\lambda-\lambda_0}d\lambda=2\pi i\sigma. \tag{26}$$

where σ is the sum of the residues of $\frac{V_1(x, \xi, \lambda)}{\lambda-\lambda_0}$. Since $\int_{C_R} \frac{V_1(x, \xi, \lambda)}{\lambda-\lambda_0} d\lambda=0$ as $R\to\infty$, we have

$$(27)\qquad \int_0^\infty \frac{V_1(x, \xi, \sqrt{\lambda})}{\lambda-\lambda_0} d\lambda - \int_0^\infty \frac{V_1(x, \xi, \sqrt{\lambda})}{\lambda-\lambda_0} d\lambda = 2\pi i \sigma.$$

The residue at the eigenvalue λ_j is

$$\lim_{\lambda\to\lambda_j} (\lambda-\lambda_j) \frac{V_1(x, \xi, \lambda)}{\lambda-\lambda_0} = \frac{y_j(x)\overline{z_j(\xi)}}{(\lambda_j-\lambda_0)\int_0^\infty y_j(\xi)\overline{z_j(\xi)}\,d\xi}.$$

Thus the sum σ of the residues is

$$(28)\qquad \sigma = \sum_{j=1}^n \frac{y_j(x)\overline{z_j(\xi)}}{(\lambda_j-\lambda_0)\int_0^\infty y_j(\xi)\overline{z_j(\xi)}\,d\xi} + V_1(x, \xi, \lambda_0).$$

Substituting (28) in (27) and simplifying it, we have

$$(29)\qquad V_1(x, \xi, \lambda_0) = \frac{1}{2\pi i}\int_0^\infty \frac{V_1(x, \xi, \sqrt{\lambda})}{\lambda-\lambda_0} d\lambda - \frac{1}{2\pi i}\int_0^\infty \frac{V_1(x, \xi, -\sqrt{\lambda})}{\lambda-\lambda_0} d\lambda$$
$$- \frac{1}{2\pi i}\sum_{j=1}^n \frac{y_j(x)\overline{z_j(\xi)}}{(\lambda_j-\lambda_0)\int_0^\infty y_j(\xi)\overline{z_j(\xi)}\,d\xi}.$$

Thus, we obtain the following theorem.

THEOREM 7. *Let $\lambda_0=s_0^2$ be not an eigenvalue of the operator L_K. Then*

$$(30)\qquad V_1(x, \xi, \lambda_0) = \frac{1}{2\pi i}\int_0^\infty \frac{V_1(x, \xi, \sqrt{\lambda})}{\lambda-\lambda_0} d\lambda - \frac{1}{2\pi i}\int_0^\infty \frac{V_1(x, \xi, -\sqrt{\lambda})}{\lambda-\lambda_0} d\lambda$$
$$- \frac{1}{2\pi i}\sum_{j=1}^n \frac{y_j(x)\overline{z_j(\xi)}}{(\lambda_j-\lambda_0)\int_0^\infty y_j(\xi)\overline{z_j(\xi)}\,d\xi}.$$

We now consider $V_2(x, \xi, \lambda)$. Let $\lambda_0=s_0^2$ be in the interior of C_R and such that $\lambda_0=s_0^2$ is not an eigenvalue of the operator L_K. If we integrate the function

$$(31)\qquad \frac{V_2(x, \xi, \lambda)}{\lambda-\lambda_0} = \frac{e^{-i\sqrt{\lambda}x_<} e^{i\sqrt{\lambda}x_>}}{2i\sqrt{\lambda}(\lambda-\lambda_0)}$$

around the contour in Figure 1, we have

$$(32)\qquad \int_{C+} \frac{V_2(x, \xi, \lambda)}{\lambda-\lambda_0} d\lambda + \int_{C_R} \frac{V_2(x, \xi, \lambda)}{\lambda-\lambda_0} d\lambda + \int_{C-} \frac{V_2(x, \xi, \lambda)}{\lambda-\lambda_0} d\lambda = 2\pi i \sigma$$

where σ is the sum of the residues of

$$\frac{V_2(x, \xi, \lambda)}{\lambda-\lambda_0}.$$

Since $\int_{C_R} \frac{V_2(x, \xi, \lambda)}{\lambda-\lambda_0} d\lambda = 0$ as $R \to \infty$, we have

$$(33) \qquad \int_0^\infty \frac{e^{-i\sqrt{\lambda}x} \langle e^{i\sqrt{\lambda}x} \rangle}{2i\sqrt{\lambda}\,(\lambda-\lambda_0)} d\lambda + \int_\infty^0 \frac{e^{i\sqrt{\lambda}x} \langle e^{-i\sqrt{\lambda}x} \rangle}{-2i\sqrt{\lambda}\,(\lambda-\lambda_0)} d\lambda = 2\pi i \sigma$$

The left hand side of (33) becomes

$$(34) \qquad \int_0^\infty \frac{e^{-i\sqrt{\lambda}x} e^{i\sqrt{\lambda}\xi} + e^{-i\sqrt{\lambda}\xi} e^{i\sqrt{\lambda}x}}{2i\sqrt{\lambda}\,(\lambda-\lambda_0)} d\lambda.$$

Since the function $\frac{V_2(x, \xi, \lambda)}{\lambda-\lambda_0}$ has no singularity except $\lambda=\lambda_0$, the sum σ of the residues is

$$(35) \qquad \sigma = \lim_{\lambda \to \lambda_0} \frac{(\lambda-\lambda_0)\, V_2(x, \xi, \lambda)}{(\lambda-\lambda_0)} = V_2(x, \xi, \lambda_0).$$

Therefore we have the following theorem.

THEOREM 8. *Let $\lambda_0 = s_0^2$ be not an eigenvalue of the operator L_K. Then*

$$(36) \qquad V_2(x, \xi, \lambda_0) = \frac{1}{2\pi i} \int_0^\infty \frac{e^{-i\sqrt{\lambda}x} e^{i\sqrt{\lambda}\xi} + e^{i\sqrt{\lambda}\xi} e^{-i\sqrt{\lambda}x}}{2i\sqrt{\lambda}\,(\lambda-\lambda_0)} d\lambda.$$

Combining theorem 7 and 8, we have the following theorem.

THEOREM 9. *Let $\lambda_0 = s_0^2$ be not an eigenvalue of the operator L_K, and $\{\lambda_k\}_{k=1}^n$ be the set of simple eigenvalues of the operator L_K. Then*

$$(37) \quad G(x, \xi, \lambda_0) = V_1(x, \xi, \lambda_0) + V_2(x, \xi, \lambda_0)$$

$$= \frac{1}{2\pi i} \int_0^\infty \frac{V_1(x, \xi, \sqrt{\lambda})}{\lambda-\lambda_0} d\lambda - \frac{1}{2\pi i} \int_0^\infty \frac{V_1(x, \xi, -\sqrt{\lambda})}{\lambda-\lambda_0} d\lambda$$

$$- \frac{1}{2\pi i} \sum_{j=1}^n \frac{y_j(x)\overline{z_j(\xi)}}{(\lambda_j-\lambda_0) \int_0^\infty y_j(\xi)\overline{z_j(\xi)}\, d\xi}$$

$$+ \frac{1}{2\pi i} \int_0^\infty \frac{e^{-i\sqrt{\lambda}x} e^{i\sqrt{\lambda}\xi} + e^{i\sqrt{\lambda}\xi} e^{-i\sqrt{\lambda}x}}{2i\sqrt{\lambda}\,(\lambda-\lambda_0)} d\lambda.$$

Consider the set D^* of those functions defined by

(1) $g(x)$ is in $L^1(0, \infty)$

(2) $g'(x)$ exists and is absolutely continuous on every finite subinterval

$[0,b]$ of $[0,\infty)$

(3) $g''(x)-h(x)(a_1g(0)+a_2g'(0))$ is in $L^1(0,\infty)$

(4) $\int_0^\infty K(x)g(x)dx=b_1g(0)-b_2g'(0)$.

If $g(x)$ is in D^*, then $\lim_{x\to\infty} g(x)=\lim_{x\to\infty} g'(x)=0$. To prove this proposition, see Krall[2]. Assume $g(x)$ is in D^* and $f(x)=g''(x)+\lambda_0g(x)-h(x)(a_1g(0)+a_2g'(0))$ and $\lambda_0=s_0^2$ is not an eigenvalue of the operator L_K. Then

$$g(x)=\int_0^\infty G(x,\xi,\lambda_0)f(\xi)d\xi. \tag{38}$$

So, $g(x)=\int_0^\infty [V_1(x,\xi,\lambda_0)+V_2(x,\xi,\lambda_0)]f(\xi)d\xi$

$$=\int_0^\infty\Big[\frac{1}{2\pi i}\int_0^\infty\frac{V_1(x,\xi,\sqrt{\lambda})}{\lambda-\lambda_0}d\lambda-\frac{1}{2\pi i}\int_0^\infty\frac{V_1(x,\xi,-\sqrt{\lambda})}{\lambda-\lambda_0}d\lambda$$

$$-\frac{1}{2\pi i}\sum_{j=1}^n\frac{y_j(x)\overline{z_j(\xi)}}{(\lambda_j-\lambda_0)\int_0^\infty y_j(\xi)\overline{z_j(\xi)}d(\xi)}$$

$$+\frac{1}{2\pi i}\int_0^\infty\frac{e^{-i\sqrt{\lambda}x}e^{i\sqrt{\lambda}\xi}+e^{i\sqrt{\lambda}x}e^{-i\sqrt{\lambda}x}}{2i\sqrt{\lambda}(\lambda-\lambda_0)}d\lambda\Big]f(\xi)d\xi$$

$$=-\frac{1}{2\pi i}\int_0^\infty g(\xi)\Big(\int_0^\infty[V_1(x,\xi,\sqrt{\lambda})-V_1(x,\xi,-\sqrt{\lambda})]d\lambda\Big)d\xi$$

$$-\frac{1}{2\pi i}\int_0^\infty g(\xi)\Big(\int_0^\infty\Big[\frac{e^{-i\sqrt{\lambda}x}e^{i\sqrt{\lambda}\xi}+e^{i\sqrt{\lambda}x}e^{-i\sqrt{\lambda}\xi}}{2i\sqrt{\lambda}}\Big]d\lambda\Big)d\xi$$

$$+\sum_{j=1}^n\frac{y_j(x)\int_0^\infty\overline{z_j(\xi)}g(\xi)d\xi}{\int_0^\infty\overline{z_j(\xi)}y_j(\xi)d\xi}$$

Therefore we have the following theorem.

THEOREM 10. *Let $g(x)$ be in D^*. Then*

$$g(x)=-\frac{1}{2\pi i}\int_0^\infty g(\xi)\Big(\int_0^\infty[V_1(x,\xi,\sqrt{\lambda})-V_1(x,\xi,-\sqrt{\lambda})]d\lambda\Big)d\xi \tag{39}$$

$$-\frac{1}{2\pi i}\int_0^\infty g(\xi)\Big(\int_0^\infty\Big[\frac{e^{-i\sqrt{\lambda}x}e^{i\sqrt{\lambda}\xi}+e^{i\sqrt{\lambda}x}e^{-i\sqrt{\lambda}\xi}}{2i\sqrt{\lambda}}\Big]d\lambda\Big)d\xi$$

$$+\sum_{j=1}^n\frac{y_j(x)\int_0^\infty\overline{z_j(\xi)}g(\xi)d\xi}{\int_0^\infty\overline{z_j(\xi)}y_j(\xi)d\xi}$$

3. Conclusion.

Using the differential expression $ly=y''-h(x)(a_1y(0)+a_2y'(0))$ and the boundary condition $\int_0^\infty K(x)y(x)dx=b_1y(0)-b_2y'(0)$, we define an operator L_K by $L_Ky=y''-h(x)(a_1y(0)+a_2y'(0))$ for all functions satisfying the above boundary condition. When $\lambda_0=s_0^2$ is not an eigenvalue, the green's function has the following form.

$$G(x,\xi,\lambda_0)=V_1(x,\xi,\lambda_0)+V_2(x,\xi,\lambda_0),$$

where

$$V_1(x,\xi,\lambda_0)=\frac{1}{2\pi i}\int_0^\infty\frac{V_1(x,\xi,\sqrt{\lambda})}{\lambda-\lambda_0}d\lambda-\frac{1}{2\pi i}\int_0^\infty\frac{V_1(x,\xi,-\sqrt{\lambda})}{\lambda-\lambda_0}d\lambda$$

$$-\frac{1}{2\pi i}\sum_{j=1}^{n}\frac{y_j(x)\overline{z_j(\xi)}}{(\lambda_j-\lambda_0)\int_0^\infty y_j(\xi)\overline{z_j(\xi)}d\xi}$$

$$V_2(x,\xi,\lambda_0)=\frac{1}{2\pi i}\int_0^\infty\frac{e^{-i\sqrt{\lambda}x}e^{i\sqrt{\lambda}\xi}+e^{-i\sqrt{\lambda}x}e^{i\sqrt{\lambda}\xi}}{2i\sqrt{\lambda}(\lambda-\lambda_0)}d\lambda.$$

The eigenfunction expansion of a certain function $g(x)$ has the form

$$g(x)=-\frac{1}{2\pi i}\int_0^\infty g(\xi)\left(\int_0^\infty[V_1(x,\xi,\sqrt{\lambda})-V_1(x,\xi,-\sqrt{\lambda})]d\lambda\right)d\xi$$

$$-\frac{1}{2\pi i}\int_0^\infty g(\xi)\left(\int_0^\infty\left[\frac{e^{-i\sqrt{\lambda}x}e^{i\sqrt{\lambda}\xi}+e^{-i\sqrt{\lambda}\xi}e^{i\sqrt{\lambda}x}}{2i\sqrt{\lambda}}\right]d\lambda\right)d\xi$$

$$+\sum_{j=1}^{n}\frac{y_j(x)\int_0^\infty\overline{z_j(\xi)}g(\xi)d\xi}{\int_0^\infty\overline{z_j(\xi)}y_j(\xi)d\xi}.$$

REMARK. If $h(x)$ is identically zero on $(0,\infty)$, the operator L_K reduces to the operator L discussed by the Krall[2]. If $h(x)$ and $K(x)$ are identically zero on the interval $(0,\infty)$, the operator L_K reduces to the operator L_θ discussed by Naimark[3]. So this paper is some extension or generlized one of Krall's and Naimark's.

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