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INFINITESIMAL VARIATIONS OF GENERIC SUBMANIFOLDS OF A MANIFOLD WITH NORMAL (f, g, u, v, λ)-STRUCTURE

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Introduction

Ki ([3], [4], [5]), Chen ([2]) and Yano ([7], [8], [9]) have recently studied infinitesimal variations of submanifolds. On the other hand, Ki and Okumura have studied infinitesimal variations of generic submanifolds of a Kaehlerian manifold.

The purpose of this paper is to study infinitesimal variations of generic submanifolds of the ambient manifold with normal (f, g, u, v, λ)-structure. In § 1, we compare some properties of generic submanifolds of the ambient manifold with normal (f, g, u, v, λ)-structure. In § 2, we prove the fundamental formulas in the theory of infinitesimal variations, that is, which carry a generic submanifold into a generic submanifold. In § 3, we study f-preserving variations and compute the variations of u, v, and λ. In § 4, we find the conditions that the variation vectors are parallel and prove complete hypersurface of an even-dimensional sphere is the product of two spheres under some conditions.

§ 1. Generic submanifolds of a manifold with normal (f, g, u, v, λ)-structure

Let M^{2m} be a real $2m$-dimensional manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$, in which a manifold with a tensor field f of type(1, 1), a Riemannian metric g, two 1-forms u, v and a function λ satisfying

$$(1.1)\qquad \begin{cases} f_j{}^t f_t{}^h = -\delta_j^h + u_j u^h + v_j v^h, \\ f_j{}^t f_i{}^s g_{ts} = g_{ji} - u_j u_i - v_j v_i, \\ u_t f_j{}^t = \lambda v_j, \quad f_t{}^h u^t = -\lambda v^h, \\ u_t u^t = v_t v^t = 1 - \lambda^2, \; u_t v^t = 0, \end{cases}$$

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f_i^h, g_{ji}, u_i, v_i and λ being respectively components of f, g, u, v and λ with respect to a local coordinate system, u^h and v^h being defined by $u_j=g_{ji}u^i$ and $v_j=g_{ji}v^i$ respectively, where here and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1, 2, 3, \cdots, 2m\}$, then the structure is called an (f, g, u, v, λ)-structure ([5], [8]). It is known that such a manifold is even-dimensional ([5]). If we put $f_{ji}=f_j^t g_{ti}$, we can easily see that f_{ji} is skew-symmetric.

We put

$$S_{ji}^h=[f,f]_{ji}^h+(\nabla_j u_i-\nabla_i u_j)u^h+(\nabla_j v_i-\nabla_i v_j)v^h,$$

$[f,f]_{ji}^h$ denoting the Nijenhuis Tensor formed with f_i^h and ∇_i the operator of covariant differentiation with respect to the Christoffel symbols Γ_{ji}^h formed with g_{ji}. If S_{ji}^h vanishes, then the (f, g, u, v, λ)-structure is said to be normal ([8]).

The following theorem is well known ([8]).

THEOREM A. *Let M^{2m} be a manifold with normal (f, g, u, v, λ)-structure satisfying $\nabla_j v_i-\nabla_i v_j=2f_{ji}$. If the function $\lambda(1-\lambda^2)$ does not vanish almost everywhere, then we have*

$$\text{(1.2)}\qquad \begin{cases} \nabla_j f_i^h=g_{ji}(\phi u^h-v^h)-\delta_j^h(\phi u_i-v_i), \\ \nabla_j u_i=-\lambda g_{ji}-\phi f_{ji}, \\ \Delta_j\lambda=u_j+\phi v_j, \end{cases}$$

ϕ being constant. Moreover, if M^{2m} is complete and dim $M^{2m}>2$, then M^{2m} is isometric with an even-dimensional sphere.

Let M^n be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V: y^a\}$ and with metric tensor g_{cb}, where here and in the sequel, the indices $a, b, c, \ldots$ run over the range $\{1, 2, \ldots, n\}$. We assume that M^n is isometrically immersed in M^{2m} by the immersion $i: M^n \to M^{2m}$ and identify $i(M^n)$ with M^n itself. We represent the immersion i locally by $x^h=x^h(y^a)$ and put $B_b^h=\partial_b x^h$, $\partial_b=\partial/\partial y^b$, which are linearly independent vectors of M^{2m} tangent to M^n. Since the immersion i is isometric, we have

$$\text{(1.3)}\qquad g_{cb}=g_{ji}B_c^j B_b^i.$$

We denote by C_y^h $2m-n$ mutually orthogonal unit normals to M^n, where here and in the sequel, the indices $x, y, z, \ldots$ run over the range $\{n+1, n+2, \ldots, 2m\}$. Then the equations of Gauss are written as

$$\text{(1.4)}\qquad \nabla_c B_b^h=h_{cb}{}^x C_x^h,$$

∇_c being the operator of van der Waerden–Bortollotti covariant differentiation along M^n and $h_{cb}{}^x$ are second fundamental tensors of M^n with respect to the normals $C_x{}^h$, and those of Weingarten as

$$\nabla_c C_x{}^h = -h^a_{cx} B_a{}^h, \tag{1.5}$$

where $h^a_{cx} = h_{cbx} g^{ba} = h^z_{cb} g_{zx}$, $(g^{ba}) = (g_{ba})^{-1}$ and g_{zx} denoting the metric tensor of the normal bundle.

If the transform by f of any vector tangent to M^n is always tangent to M^n, that is, if there exists a tensor field $f_b{}^a$ of type (1, 1) such that

$$f_i{}^h B_b{}^i = f_b{}^a B_a{}^h - f_b{}^x C_x{}^h, \tag{1.6}$$

we say that M^n is *generic* in M^{2m}.

For the transform by $f_i{}^h$ of normal vectors $C_y{}^i$, we have equations of the form

$$f_i{}^h C_y{}^i = f_y{}^a B_a{}^h \tag{1.7}$$

where $f_y{}^h = f_b{}^x g^{ba} g_{yx}$, which can also be written as $f_{ya} = -f_{ay}$. We put

$$\begin{cases} u^h = u^a B_a{}^h + u^x C_x{}^h, \\ v^h = v^a B_a{}^h + v^x C_x{}^h, \end{cases} \tag{1.8}$$

u^a and v^a being vector fields of M^n, u^x and v^x being functions of M^n.

From (1.1), (1.6), (1.7) and (1.8), we find

$$f_b{}^c f_c{}^a - f_b{}^x f_x{}^a = -\delta_b^a + u_b u^a + v_b v^a, \tag{1.9}$$

$$f_b{}^a f_a{}^x = -u_b u^x - v_b v^x, \tag{1.10}$$

$$f_y{}^a f_a{}^x = \delta_y^x - u_y u^x - v_y v^x, \tag{1.11}$$

$$u^b f_b{}^a + u^x f_x{}^a = -\lambda v^a, \quad v^b f_b{}^a + v^x f_x{}^a = \lambda u^a, \tag{1.12}$$

$$u^a f_a{}^x = \lambda v^x, \quad v^a f_a{}^x = -\lambda u^x, \tag{1.13}$$

$$u_a u^a + u_x u^x = 1 - \lambda^2, \quad v_a v^a + v_x v^x = 1 - \lambda^2, \tag{1.14}$$

$$u_a v^a + u_x v^x = 0. \tag{1.15}$$

We also have from (1.6), $f_{ji} B_c{}^j B_b{}^i = f_c{}^a g_{ba}$. Thus, putting $f_c{}^a g_{ba} = f_{cb}$, we see that f_{cb} is skew-symmetric.

Differentiating (1.6) and (1.7) covariantly along M^n, and using (1.2), (1.4) and (1.5), we find

$$\nabla_c f_b{}^a = g_{cb}(\phi u^a - v^a) - \delta_c^a(\phi u_b - v_b) - h_{cb}{}^x f_x{}^a - h_{cx}{}^a f_b{}^x, \tag{1.16}$$

$$\nabla_c f_b{}^x = f_b{}^a h_{ca}{}^x - g_{cb}(\phi u^x - v^x), \tag{1.17}$$

(1.18) $$\nabla_c f_y{}^a = -\delta_c{}^a(\phi u_y - v_y) - h_{cy}{}^b f_b{}^a,$$

(1.19) $$f_y{}^a h_{ca}{}^x = h_{cy}{}^a f_a{}^x.$$

On the other hand, differentiating u^h, v^h covariantly along M^n and using (1.2), (1.6) and (1.4), we get

(1.20) $$\nabla_c u^a = u^x h_{cx}{}^a - \lambda \delta_c{}^a - \phi f_c{}^a,$$

(1.21) $$\nabla_c u^x = -u^a h_{ca}{}^x + \phi f_c{}^x,$$

(1.22) $$\nabla_c v^a = v^x h_{cx}{}^a - \phi\lambda\delta_c{}^a + f_c{}^a,$$

(1.23) $$\nabla_c v^x = -v^a h_{ca}{}^x - f_c{}^x,$$

(1.24) $$\nabla_c \lambda = u_c + \phi v_c.$$

§2. Infinitesimal variations of generic submanifolds

We consider an infinitesimal variation of generic submanifold M^n of a manifold M^{2m} with normal (f, g, u, v, λ)-structure given by

(2.1) $$\bar{x}^h = x^h(y) + \xi^h(y)\varepsilon,$$

where $\xi^h(y)$ is a vector field of M^{2m} defined along M^n and ε is an infinitesimal. We then have

(2.2) $$\bar{B}_b{}^h = B_b{}^h + (\partial_b \xi^h)\varepsilon,$$

where $\bar{B}_b{}^h = \partial_b \bar{x}^h$ are n linearly independent vectors tangent to the varied submanifold. We displace $\bar{B}_b{}^h$ parallelly from the varied point $(\bar{x}^h)$ to the original point (x^h). We then obtain the vectors

$$\tilde{B}_b{}^h = \bar{B}_b{}^h + \Gamma_{ji}{}^h(x+\xi\varepsilon)\xi^j \bar{B}_b{}^i \varepsilon$$

at the point (x^h), or

(2.3) $$\tilde{B}_b{}^h = B_b{}^h + (\nabla_b \xi^h)\varepsilon,$$

neglecting the terms of order higher than one with respect to ε, where

(2.4) $$\nabla_b \xi^h = \partial_b \xi^h + \Gamma_{ji}{}^h B_b{}^j \xi^i.$$

In the sequel we always neglect terms of order higher than one with respect to ε. Thus, putting

(2.5) $$\delta B_b{}^h = \tilde{B}_b{}^h - B_b{}^h,$$

we have from (2.3)

(2.6) $$\delta B_b{}^h = (\nabla_b \xi^h)\varepsilon.$$

Putting

(2.7) $$\xi^h=\xi^a B^h{}_a+\xi^x C_x{}^h,$$

we have

(2.8) $$\nabla_b\xi^h=(\nabla_b\xi^a-h_{bx}{}^a\xi^x)B_a{}^h+(\nabla_b\xi^x+h_{ba}{}^x\xi^a)C_x{}^h.$$

Now we denote by $\bar{C}_y{}^h$ $2m-n$ mutually orthogonal unit normals to the varied submanifold and $\tilde{C}_c{}^h$ the vectors obtained from $\bar{C}_y{}^h$ by parallel displacement of $\bar{C}_y{}^h$ from the point $(\bar{x}^h)$ to (x^h). Then we have

(2.9) $$\tilde{C}_y{}^h=\bar{C}_y{}^h+\Gamma_{ji}{}^h(x+\xi\varepsilon)\xi^j\bar{C}_j{}^i\varepsilon.$$

We put

(2.10) $$\delta C_y{}^h=\tilde{C}_y{}^h-C_y{}^h$$

and assume that $\delta\, C_y{}^h$ is of the form

(2.11) $$\delta C_y{}^h=\eta_y{}^h\varepsilon=(\eta_y{}^a B_a{}^h+\eta_y{}^x C_x{}^h)\varepsilon.$$

Then, from (2.6), (2.10) and (2.11), we have

(2.12) $$\bar{C}_y{}^h=C_y{}^h-\Gamma_{ji}{}^h\xi^j C_y{}^i\varepsilon+(\eta_y{}^a B_a{}^h+\eta_y{}^x C_x{}^h)\varepsilon.$$

A pplying the operator δ to $B_b{}^j C_y{}^i g_{ji}=0$ and using (2.7), (2.8), (2.11) and $\delta\, g_{ji}=0$, we find

(2.13) $$(\nabla_b\xi_y+h_{ba}{}_y\xi^a)+\eta_{yb}=0,$$

where $\xi_y=\xi^z g_{zy}$ and $\eta_{yb}=\eta_y{}^c g_{cb}$, or

(2.14) $$\eta_y{}^a=-(\nabla^a\xi_y+h_{by}{}^a\xi^b),$$

∇^a being defined to be $\nabla^a=g^{ac}\nabla_c$. Applying the operator δ to $C_y{}^j C_x{}^i g_{ji}=\delta_{yx}$ and using (2.11) and $\delta\, g_{ji}=0$, we find

(2.15) $$\eta_{yx}+\eta_{xy}=0,$$

where $\eta_{yx}=\eta_y{}^z g_{zx}$.

Suppose that an infinitesimal variation given by (2.1) carries a submanifold into another submanifold and the tangent space of the original submanifold at a point and that of the varied submanifold at the corresponding point are parallel. Then we say that such a variation is ***parallel*** ([7]).

We assume that the infinitesimal variation (2.1) carries a generic submanifold into a generic submanifold, that is,

(2.16) $f_i{}^h(x+\xi\varepsilon)\bar{C}_y{}^i$ *are linear combination of* $\bar{B}_b{}^h$.

Using the first equation (1.2), (1.6) and (2.12), we see that

$$f_i{}^h(x+\xi\varepsilon)\bar{C}_y{}^i=(f_i{}^h+\xi^j\partial_j f_i{}^h\varepsilon)(C_y{}^i-\Gamma_{jt}{}^i\xi^j C_y{}^t\varepsilon+[\eta_y{}^a B_a{}^i+\eta_y{}^x C_x{}^i]\varepsilon)$$

$$=f_i{}^hC_y{}^i-\Gamma_{jt}{}^i\xi^jC_y{}^tf_i{}^h\varepsilon+\eta_y{}^af_i{}^hB_a{}^i\varepsilon+\eta_y{}^xf_i{}^hC_x{}^i\varepsilon+\xi^j(\partial_jf_i{}^h)C_y{}^i\varepsilon,$$

which and (2.12) imply

$$f_i{}^h(x+\xi\varepsilon)\bar{C}_y{}^i$$
$$=f_y{}^a+[(\eta_y{}^bf_b{}^a+\eta_y{}^xf_x{}^a)B_a{}^h-\eta_y{}^af_a{}^xC_x{}^h-f_y{}^a(\partial_a\xi^h)$$
$$-\Gamma_{jk}{}^h\xi^jf_y{}^aB_a{}^k+\xi_y(\phi u^h-v^h)-\xi^h(\phi u_y-v_y)]\varepsilon, \tag{2.17}$$

or, using (1.6), (1.7), (1.8), (2.7) and (2.8),

$$f_i{}^h(x+\xi\varepsilon)\bar{C}_y{}^i=f_y{}^a\bar{B}_a{}^h$$
$$+[\eta_y{}^xf_x{}^a-f_b{}^a(\nabla^b\xi_y+h_{ey}{}^b\xi^e)-f_y{}^b(\nabla_b\xi^a-h_{bx}{}^a\xi^x)$$
$$+\xi_y(\phi u^a-v^a)-\xi^a(\phi u_y-v_y)]B_a{}^h\varepsilon$$
$$+[f_a{}^x(\nabla^a\xi_y+h_{ey}{}^a\xi^e)-f_y{}^b(\nabla_b\xi^x+h_{ba}{}^x\xi^a)$$
$$+\xi_y(\phi u^x-v^x)-\xi^x(\phi u_y-v_y)]C_x{}^h\varepsilon. \tag{2.18}$$

Thus, we can see that (2.16) is equivalent to

$$\xi_y(\phi u^x-v^x)-\xi^x(\phi u_y-v_y)$$
$$=f_y{}^a(\nabla_a\xi^x+h_{ba}{}^x\xi^b)-f_b{}^x(\nabla^b\xi_y+h_{ay}{}^b\xi^a). \tag{2.19}$$

An infinitesimal variation given by (2.1) is called a *generic-preserving* variation if it carries a generic submanifold into a generic submanifold. Thus, we have

THEOREM 2.1. *In order for an infinitesimal variation of a manifold with normal (f, g, u, v, λ)-structure to be a generic-preserving, it is necessary and sufficient that the variation* (2.1) *satisfies* (2.19).

COROLLARY 2.2. *In order for an infinitesimal variation of an even-dimensional sphere to be a generic-preserving, it is necessary and sufficient that the variation* (2.1) *satisfies*

$$\xi^xv_y-\xi_yv^x=f_y{}^a(\nabla_a\xi^x+h_{ba}{}^x\xi^b)-f_b{}^x(\nabla^b\xi_y+h_{ay}{}^b\xi^a) \tag{2.20}$$

THEOREM 2.3. *If an infinitesimal variation of the submanifold of the manifold M^{2m} with normal (f, g, u, v, λ)-structure is normal and u^h, v^h are tangent to the submanifold M^n. Then the variation is a generic-preserving.*

COLOLLARY 2.4. *If an infinitesimal variation of the submanifold of an even-dimensional sphere S^{2m} is parallel and v^h is tangent to the submanifold M^n. Then the variation is a generic-preserving.*

§ 3. The variation of structure

Suppose that an infinitesimal variation $\bar{x}^h = x^h + \xi^h \varepsilon$ is a generic-preserving variation. Then putting

$$f_i^h(x+\xi\varepsilon)\bar{B}_b^i = (f_b^a + \delta f_b^a)\bar{B}_a^h - (f_b^x + \delta f_b^x)\bar{C}_x^h, \tag{3.1}$$

we have from (1.6), (1.7), (2.6) and (2.7)

$$\begin{aligned}\delta f_b^a = [&f_e^a(\nabla_b\xi^e - h_{bx}{}^e\xi^x) - f_b^e(\nabla_e\xi^a - h_{ex}{}^a\xi^x)\\ &+ f_b^a(\nabla_b\xi^x + h_{bc}{}^x\xi^c) - f_b^x(\nabla^a\xi_x + h_{ex}{}^a\xi^e)\\ &+ \xi_b(\phi u^a - v^a) + \xi^a(\phi u_b - v_b)]\varepsilon\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\delta f_b^x = [&f_b^y\eta_y{}^x - f_a^x(\nabla_b\xi^a - h_{by}{}^a\xi^y) - f_b^a(\nabla_a\xi^x + h_{ae}{}^x\xi^e)\\ &+ \xi_b(\phi u^x - v^x) - \xi^x(\phi u_b - v_b)]\varepsilon.\end{aligned} \tag{3.3}$$

If a generic-preserving variation preserves f_b^a and b_b^x, then we say that it is *f-preserving.*

PROPOSITION 3. 1. *A generic-preserving variation is f-preserving if and only if the brackets of* (3. 2) *and* (3. 3) *vanish.*

Now applying the operator δ to (1. 3) and using (2. 6), (2. 8) and $\delta\, g_{ji} = 0$, wefind ([7])

$$\delta g_{cb} = (\nabla_c\xi_b + \nabla_b\xi_c - 2h_{cbx}\xi^x)\varepsilon, \tag{3.4}$$

from which,

$$\delta g^{ba} = -(\nabla^b\xi^a + \nabla^a\xi^b - 2h_x{}^{ba}\xi^x)\varepsilon. \tag{3.5}$$

Assume that an infinitesimal variation $\bar{x}^h = x^h + \xi^h\varepsilon$ is generic-preserving. Hence, we have

$$f_i^h\bar{C}_y{}^i = f_y{}^a\bar{B}_a{}^h. \tag{3.6}$$

From (2. 18), we obtain

$$\begin{aligned}\delta f_y{}^i = [&f_x{}^a\eta_y{}^x - f_b{}^a(\nabla^b\xi_y + he_y{}^b\xi^e) - f_y{}^b(\nabla_b\xi^a - h_{bx}{}^a\xi^x)\\ &+ \xi_y(\phi u^a - v^a) - \xi^a(\phi u_y - v_y)]\varepsilon.\end{aligned} \tag{3.7}$$

PROPOSITION 3. 2. *Suppose that an infinitesimal variation is a generic-preserving. Then the variation of $f_y{}^a$ is given by* (3. 7).

Now we get a vector field $\bar{u}^h$ which is defined intrinsically along the deformed submanifold. If we deplace $\bar{u}^h$ back parallelly from the point $(\bar{x}^h)$ to (x^h), we obtain

$$\tilde{u}^h = \bar{u}^h + \Gamma_{ji}{}^h(x+\xi\varepsilon)\xi^j\bar{u}^i,$$

and hence forming

$$\delta u^h = \tilde{u}^h - \bar{u}^h$$

we find

$$\delta u^h = \bar{u}^h - u^h + \Gamma_{ji}{}^h \xi^j u^i \varepsilon. \tag{3.8}$$

Further we have, from (1.8) and (3.8),

$$\delta(u^a B_a{}^h + u^x C_x{}^h) = \xi^k(\partial_k u^h)\varepsilon + \Gamma_{ji}{}^h \xi^j u^i \varepsilon, \tag{3.9}$$

which and (2.6), (2.7) and (2.12) imply

$$\delta u^a = -[u^b(\nabla_b \xi^a - h_{bx}{}^a \xi^x) - u^y(\nabla^a \xi_y + h_{by}{}^a \xi^b) + \lambda\xi^a + \phi\xi^b f_b{}^a + \phi\xi^x f_x{}^a]\varepsilon \tag{3.10}$$

and

$$\delta u^x = -[u^b(\nabla_b \xi^x + h_{ba}{}^x \xi^a) + u^y \eta_y{}^x + \lambda\xi^x + \phi\xi^b b_b{}^x]\varepsilon. \tag{3.11}$$

From which, using (3.4),

$$\delta u_b = -[u^e(-\nabla_b \xi_e + h_{bex}\xi^x) - u^y(\nabla_b \xi_y + h_{eby}\xi^e) + \lambda\xi_b + \phi\xi^e f_{eb} + \phi\xi^x f_{xb}]\varepsilon. \tag{3.12}$$

and

$$\delta u_y = -[u^b(\nabla_b \xi_y + h_{bey}\xi^e) + u^x \eta_{xy} + \lambda\xi_y + \phi\xi^b f_{by}]\varepsilon. \tag{3.13}$$

Thus we have

PROPOSITION 3.3. *Under an infinitesimal variation* (2.1) *of the submanifold, the variations of* u^a, u^x, u_b *and* u_y *are given by* (3.10), (3.11), (3.12) *and* (3.13) *respectively.*

Similarly we get a vector field $\bar{v}^h$ which is defined intrinsically from the point $(\bar{x}^h)$ to (x^h), we have

$$\tilde{v}^h = \bar{v}^h + \Gamma_{ji}{}^h(x+\xi\varepsilon)\xi^j\bar{v}^i\varepsilon$$

and hence forming

$$\delta v^h = \tilde{v}^h - v^h, \tag{3.14}$$

we find

$$\delta v^h = \bar{v}^h - v^h + \Gamma_{ji}{}^h \xi^j v^i \varepsilon. \tag{3.15}$$

We have, from (1.8) and (3.15),

(3.16) $$\delta(v^a B_a{}^h + v^x C_x{}^h) = \xi^k(\partial_k v^h)\varepsilon + \Gamma_{ji}{}^h v^i \varepsilon,$$

which and (1.2), (2.6) and (2.16) imply

(3.17) $$\delta v^a = -[v^b(\nabla_b \xi^a - h_{bx}{}^a \xi^x) - v^y(\nabla^a \xi_y + h_{by}{}^a \xi^b) + \phi\lambda\xi^a - \xi^b f_b{}^a - \xi^y f_y{}^a]\varepsilon$$

and

(3.18) $$\delta v^x = -[v^b(\nabla_b \xi^x + h_{ba}{}^x \xi^a) + v^y \eta_y{}^x + \xi^b f_b{}^x + \phi\lambda\xi^x]\varepsilon.$$

From which, using (3.4), we obtain

(3.19) $$\delta v_b = [v^e(\nabla_b \xi_e - h_{bex}\xi^x) - v^y(\nabla_b \xi_y + h_{eby}\xi^e) + \phi\lambda\xi_b - \xi^e f_{eb} - \xi^y f_{yb}]\varepsilon$$

and

(3.20) $$\delta v_y = -[v^b(\nabla_b \xi_y + h_{bay}\xi^a) + v^x \eta_{xy} + \xi^b f_{by} + \phi\lambda\xi_y]\varepsilon.$$

Thus, we have

PROPOSITION 3.4. *Under an infinitesimal variation* (2.1) *of the submanifold, the variation of* v^a, v^x, v_b *and* v_y *are given by* (3.17), (3.18), (3.19) *and* (3.20) *respectively.*

Finally, to obtain the variation of λ, applying the operator δ to $u^a u_a + u^x u_x = 1 - \lambda^2$ and using (3.10), (3.11), (3.12) and (3.13), we obtain

(3.21) $$\delta\lambda = [(u_a + \phi v_a)\xi^a + (u_x + \phi v_x)]\varepsilon.$$

Thus we have

PROPOSITION 3.5. *Under an infinitesimal variation* (2.1) *of the submanifold, the variation of* λ *is given by* (3.21).

§4. Infinitesimal generic variation preserving $f_b{}^a$

In this section we only consider that an infinitesimal generic variation (2.1) satisfying (2.20) of a submanifold M^n of an even-dimensional sphere S^{2m}. Moreover we suppose that this variation is normal and preserving $f_b{}^a$. Then we have from (3.2)

(4.1) $$(h_{cex} f_b{}^e + h_{bex} f_c{}^e)\xi^x + f_{cx}\nabla_b \xi^x - f_{bx}\nabla_c \xi^x = 0,$$

from which

(4.2) $$f_{cx}\nabla_b \xi^x - f_{bx}\nabla_c \xi^x = 0$$

and

(4. 3) $$(h_{cex}f_b{}^e+h_{bex}f_c{}^e)\xi^x=0.$$

If the vectors u^h and v^h are tangent to the submanifold, then we have from (1. 10), (1. 11) and (1. 13)

(4. 4) $$f_y{}^b f_b{}^a=0,$$

(4. 5) $$f_y{}^a f_a{}^x=\delta_y^x,$$

and

(4. 6) $$u^a f_a{}^x=0,\quad v^a f_a{}^x=0,$$

respectively. Transvecting (4. 2) with u^c and $f_b{}^z$, we have

(4. 7) $$u^a \nabla_a \xi^x=0 \quad \text{or} \quad v^c \nabla_c \xi^x=0.$$

Transvecting (4. 2) with $f_a{}^c$ and $f_z{}^b$, we have

(4. 8) $$f_b{}^a f_a{}^c \nabla_c \xi^x=0.$$

Substituting (1. 9) and (4. 7) into (4. 8), we obtain $\nabla_c\xi^x=0$. Hence we have

THEOREM 4. 1. *Suppose that an infinitesimal variation of the submanifold M^n of an even-dimensional sphere S^{2m} preserving $f_b{}^a$ and is normal. If the vectors u^h and v^h are tangent to M^n, then the variation is parallel.*

We suppose that ξ^x are $(2m-n)$ linearly independent normal vectors. Then we obtain from (4. 3)

(4. 9) $$h_{ce}{}^x f_b{}^e+h_{be}{}^x f_c{}^e=0.$$

When the submanifold M^n is a hypersurface of S^{2m}, then (1. 9)$\sim$(1. 15) become to so-called $(f,\ g,\ u_{(k)},\ \alpha_{(k)})$-structure ([4]). In this case the following theorem is well-known ([4]).

THEOREM B. *Let M^{2m-1} be a hypersurface with the induced normal $(f, g, u_{(k)}, \alpha_{(k)})$-structure of a sphere s^{2m}. If $\lambda^2+(u_1{}^*)^2+(v_1{}^*)^2\neq 1$* (a. e.) *and $\lambda\neq v$* (a. e.), *then M^{2m-1} is product of two spheres, where $u_1{}^*=u^x=u_x$, $v_1{}^*=v^x=v_x$.*

From Theorem B. and (4. 9), we have

THEOREM 4. 5. *Let M^{2m-1} be a complete hypersurface of an even-dimensional sphere S^{2m} and an infinitesimal variation preserves $f_b{}^a$ and is normal. If*

$$\lambda^2+(u_1{}^*)^2+(v_1{}^*)^2-1 \quad \textit{and} \quad \lambda$$

do not vanish almost everywhere, then we have $M^{2m-1}=S^r\times S^{2m-1-r}$, where S^r is r-dimensional sphere.

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