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ON NONLINEARITY

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Many problems arising naturally in differential geometry and physics involve the study of nonlinear differential equations. While linear or algebraic methods are still powerful and important in many cases, we might claim the fundamental things that come naturally in "Nature" (in both mathematical and physical sense) are nonlinear. In this paper, we shall discuss a class of problems involving nonlinearity in differential geometry and mathematical physics. Especially we want to discuss minimal surfaces, deformation of complex structure and metrics with prescribed curvature properties in differential geometry, and Yang-Mills equations and fluid mechanics in mathematical physics.

1. Nonlinearity in differential geometry.

The simplest in this case is the one about geodesics on manifolds. Let (M^n, g) be an n-dimensional manifold with the Riemannian metric

$$ds^2=\sum_{i,j=1}^{n} g_{ij}(x)\,dx_i dx_j$$

then the geodesics on (M^n, g) can be found as the solutions of nonlinear system

$$x_i+\sum_{j,k}\Gamma_{jk}{}^{i}x_j x_k=0 \quad (i, j, k=1, 2, \cdots, n)$$

Then local existence is guaranteed by the corresponding theorem in ordinary differential equations.

Now let us discuss the next simplest problem, that is the celebrated plateau's problem [2, 3, 4]. Plateau was a physicist about one hundred fifty years ago. He studied empirically various properties of soap films. Every physicist belives that soap film is the area minimizing surface with given boundary in euclidean 3-space. But mathematically existence of soap film is itself a fundamental problem. This problem is called plateau problem. More precisely, we seek a smooth, simply connected parametric surface S in R^3 spanning the given Jordan curve Γ such that the area of S is minimal. The existence of such surface corresponds to the existence of a limit of a Cauchy sequence when we apply approximating procedure. Note that in general Cauchy sequence need not have a limit. Of course the existence of a limit in our situation is far more subtle and difficult. This problem is solved by Douglas and Rado around 1930. The importance of this problem is well appreciated by giving Douglas the first Fields medal in 1932. Let us explain this problem more concretely. Let Ω be the open unit disk in R^3. We seek a vector $r(x,y)=(u_1(x,y),\ u_2(x,y),\ u_3(x,y))$ that represents a surface S spanning Γ in such a way that (1) $\partial\Omega$ is continuously mapped onto Γ in a one to one manner, (2) the area of S

$$A(S)=\iint_\Omega(|J(u_1,u_2)|^2+-|J(u_2,u_3)|^2+|J(u_1,u_3)|^2)^{1/2}dxdy$$

is minimized, where $|J(u,v)|$ is the Jacobian determinant of u and v with respect to x, y. This can be simplified by the following arguments. For any surface $S=\{r|r=u_1(x,y),\ u_2(x,y),\ u_3(x,y)\}$ we write the first fundamental form as

$$ds^2=g_{11}dx^2+2g_{12}dxdy+g_{22}dy^2$$

where $g_{11}=r_x\cdot r_x$, $g_{12}=r_x\cdot r_y$, $g_{22}=r_y\cdot r_y$. Then

$$A(S)=\iint_\Omega(g_{11}g_{22}-g_{12}^2)^{1/2}dxdy$$

Using the inequality $\sqrt{\alpha r-\beta^2}\leq\sqrt{\alpha r}\leq 1/2\{\alpha+r\}$ with equality holding if and only if $\alpha=r$, $\beta=0$, we find

$$A(S)\leq 1/2\iint_\Omega(g_{11}+g_{22})\,dxdy$$

with equality holding if and only if $g_{12}=0$ and $g_{11}=g_{22}$. Such a first fundamental form is guaranteed by the existence of an isothermal parameters on 2-dimensional manifold. Hence in isothermal coordinates,

$$A(S)=1/2\iint_\Omega(g_{11}+g_{22})\,dxdy\ =1/2\iint_\Omega(|r_x|^2+|r_y|^2)\,dxdy.$$

That is, $A(S)$ is minimized by minimizing the Dirichlet integral

$$\iint_\Omega\{|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_3|^2\}\,dxdy$$

over all vectors $r=(u_1,u_2,u_3)$ that satisfy the boundary condition (1) on $\partial\Omega$. The corresponding Euler-Lagrange equation is simply $\Delta u_1=\Delta u_2=\Delta u_3=0$. Hence the Plateau's problem is to find harmonic vector $r=(u_1,u_2,u_3)$ satisfying nonlinear boundary condition

$$|r_x|^2=|r_y|^2 \quad\text{and}\quad r_x\cdot r_y=0.$$

An important distinction between geodesics and minimal surfaces is the following observation. The length of a rectifiable curve can be found by approximating r by sufficiently small straight line segments. However the area of a surface S cannot necessarily be found by approximating S by polyhedra.

Next let us discuss deformations of complex structure [1]. Deformations of the complex structure of a Riemann surface M were first studied by Riemann. He found that the number $m(M)$ of independent complex parameters on which the deformation depends can be completely described in terms of the genus. And this number $m(M)$ is called the number of moduli. For higher dimensional complex manifolds M^n, the analogous deformation problem is less well understood and is highly nonlinear. Existence of versal deformation of complex structure of a compact complex manifold is solved by Kuranishi 1956. Let us explain Kuranishi's theory to illustrate the nonlinear character. Let M be given a complex structure V_0 with suitable distinguished complex coordinates $z_1, \ldots, z_n$. Let $\tilde{V}$ be another complex structure underlying M with local coordinates $y_1, \ldots, y_n$ which is "near" to V_0. Then $dy_j = dz_j + \Sigma \phi_{kj} d\bar{z}_k$ with ϕ_{kj} small.

$\tilde{V}$ is called an almost complex structure near the complex structure V_0. $\tilde{V}$ will be a true complex structure on M^n if and only if the $T'M$ valued (0, 1) form

$$\omega = \Sigma \phi_{kj} (\partial / \partial z_j) d\bar{z}_k$$

satisfy an integrability condition. This condition is highly nonlinear and given by $\partial\omega = [\omega, \omega]$, where

$$\partial (a d\bar{z}^{s_1} \wedge \cdots \wedge d\bar{z}^{s_q}) = \sum_k \frac{\partial a}{\partial \bar{z}^k} d\bar{z}_k \wedge d\bar{z}^{s_1} \wedge \cdots \wedge d\bar{z}^{s_q}$$

and

$$[\phi, \psi] = \Sigma [\phi^{\alpha_1 \cdots \alpha_p}, \psi^{\beta_1 \cdots \beta_q}] d\bar{z}_{\alpha_1} \wedge \cdots \wedge d\bar{z}_p \wedge d\bar{z}_{\beta_1} \wedge \cdots \wedge d\bar{z}_{\beta_q}$$

where

$$\phi = \Sigma \phi^{\alpha_1 \cdots \alpha_p} d\bar{z}_{\alpha_1} \wedge \cdots \wedge d\bar{z}_{\alpha_p}, \quad \psi = \Sigma \psi^{\beta_1 \cdots \beta_q} d\bar{z}_{\beta_1} \wedge \cdots \wedge d\bar{z}_{\beta_q},$$

$\phi^{\alpha_1 \cdots \alpha_p}$ (resp. $\psi^{\beta_1 \cdots \beta_q}$) is skew symmetric in $\alpha_1, \cdots, \alpha_p$ and is a section of holomorphic tangent bundle $T'M$, and if $L = \Sigma \xi_\alpha(z) \varrho / \partial z_\alpha$, $L' = \Sigma \eta_\alpha(z) \partial / \partial z_\alpha$ are two cross section of $T'M$, then

$$[L, L'] = \sum_\alpha \sum_\beta \frac{\partial \eta_\alpha}{\alpha z_\beta} \xi_\beta(z) - \frac{\partial \xi_\alpha}{\partial z_\beta} \eta_\beta(z)) \frac{\partial}{\partial z_\alpha}.$$

Note that in complex 1-dim case, $[\omega, \omega] = 0$. Finally in this section, we discuss about metrics with prescribed curvature. This problem is very important and very difficult. But recently this field becomes one of the very exciting and main streams of Mathematics. (The author would like to say "the most") This phenomena is mostly due to S. T. Yau at the Institute for Advanced Study [5, 6].

The natural differential operators that come in this problem are the Monge–Ampere operator $L(\phi)=\det\left(\frac{\partial^2\phi}{\partial x^i\partial x^j}\right)$ and the complex Monge–Ampere operator $L(\phi)=\det\left(\frac{\partial^2\phi}{\partial z_i\partial \bar{z}_j}\right)$.

Another which is closely related to the above problem is the Einstein field equation. If $\Sigma g_{ij}dx^i dx^j$ is the Lorentz metric to be determined, then the operator involved in the Einstein field equation is $L(g_{ij})=R_{ij}-(R/2)g_{ij}$, where R_{ij} is the Ricci tensor and R is the scalar curvature of the Lorentz metric. From the form of these differential operators, we can see how highly nonlinear the problem is.

To explain the problems more precisely let us recall some definitions. From the curvature tensor, one can derive the following quantities. Given a point in the manifold and a two dimensional plane in the tangent space at the given point, we can form the sectional curvature of the manifold at this plane. Given a point and a tangent at a point, we can form the Ricci curvature in this tangent direction by averaging all the sectional curvatures of the two dimensional tangent planes that contains this tangent. Given a point, we can form the scalar curvature at this point by simply averaging all the sectional curvatures at this point. It is clear from these definitions that the sectional curvatures give much more information than the others.

The simplest problem concerning the scalar curvature is to deform a metric conformally to one with constant scalar curvature. The equation that is involved in such a process is

$$\Delta u=\frac{(n-2)}{4(n-1)}Ru-\frac{(n-2)}{4(n-1)}\bar{R}u(n+2)/(n-2)$$

where n is the dimension of the manifold, R and $\bar{R}$ are scalar curvatures of the undeformed and deformed metrics respectively. Due to the works of Yamabe, Trudinger, Aubin, Berger, Eliason, Kazdan–Waner, Nirenberg, Morser, Greene and Wu, we can conclude that in higher dimensions, that is greater than 25 existence of complete metrics with negative scalar curvature poses no topological restriction on the manifold. However, complete metric with nonnegative scalar curvature do require topological restriction. Lichnerowicz proved that for a compact spin manifold with positive scalar curvature, there are no harmonic spinors. Using the Atiyah–Singer index theorem, he proves that for a compact spin manifold with positive scalar curvature, the $\hat{A}$-genus is zero. Also Hitchin observed that the KO-theory invariant is zero for a compact spin manifold with positive scalar curvature.

The following is very formidable unsolved problem. Find a nice criterion for a manifold to admit a metric with positive scalar curvature. The auther likes to say that the above problem is very closely related to positive mass conjecture in general relativity which is solved by *R.* Schoen and Yau himself.

Now let us discuss very glorifying results concerning with Ricci curvature of S. T. Yau. Since the Ricci curvature is given by a tensor and the integrability condition is stronger, the problem of existence is considerably harder. The known integrability conditions are not complete, but we have outstanding and beautiful results on the problem when the manifold concerned is Kähler manifold. To explain the results recall some definitions and basic resic results again. Let $\Sigma g_{ij}dz^i \otimes d\bar{z}^j$ be a Kähler metric defined on a compact complex manifold. i. e. $\bar{g}_{ij}=g_{ji}$, and the associated (1. 1) form $\omega=\sqrt{-1}\,\Sigma g_{ij}dz^i\wedge d\bar{z}^j$ is closed. Then the (1. 1) form

$$\frac{\sqrt{-1}}{2\pi}\Sigma\frac{\partial^2}{\partial z^r\partial\bar{z}^s}\log(\det(g_{ij}))\,dz^r\wedge d\bar{z}^s$$

is closed, globally defined on the manifold and represents the first Chern class. Also this (1, 1) form is the Ricci form of the Kähler metric. Hence for a (1, 1) form to be the Ricci form of some Kähler metric, it must be closed and represents the first Chern class. Calabi conjectured that this is the only integrability condition, that is, if a closed (1, 1) form $\sum R_{ij}dz^i\wedge d\bar{z}^j$ represents the first Chern class of the given complex manifold, then there exists a Kähler metric on it whose Ricci form is the given closed (1, 1) form $\Sigma R_{ij}dz^i\wedge d\bar{z}^j$.

The equation that is needed to solve Calabi's conjecture is of the following form

$$\det\left(g_{ij}+\frac{\partial^2\phi}{\partial z^i\partial\bar{z}^j}\right)=e^F\det(g_{ij})$$

where ϕ is the unknown function and F is a smooth function so that $\int_M e^F$ is the volume of M. We require $g_{ij}+\frac{\partial^2\phi}{\partial z^i\partial\bar{z}^j}$ to be positive definite metric.

The existence and the uniqueness of the above equation is solved by S. T. Yau. He used the continuity method to prove this fact. The basic steps in the proof are giving the a priori estimates up to the third derivatives. The essential difficulty lies in the estimate of $\sup|\phi|$. The affirmative answer to Calabi's conjecture gives quite a lot of unexpected application in algebraic geometry. One of them is the uniqueness of the complex structure of the complex projective plane. Since the first Chern class of K-3 space is zero, hence we can give Ricci flat Kähler metric to it. Note that the simple connectivity of the K-3 surface guarantees that it does not admit any flat metric. For more details and results, consult the references in bibliography.

2. Nonlinearity in mathematical physics.

In recent years, there are tremendous amount of research on Yang-Mills equations among both mathematicians and therectical physicists [1, 2, 3, 6]. Physicists belive that Yang-Mills theory is the most subtle and beautiful synthesis of various forces in nature. Weinberg and Salam achieved Nobel prizes through their work in this direction. Mathematicians came to have interest in this part of mathematics (or rather say physics) because of their holomorphic structure (some thing which is algebraic or holomorphic is alway exciting stuff to mathematicians. For example, complex structure on a given manifold. see Section 1.) Consider a principal G-bundle $P(M, G)$ over manifold M. Then Yang-Mills equations are the variational equations for the norm square $|F|^2$ of the curvature F of a connection A on P. In physics terminology, $|F|^2$ is the action, F the gauge field, A the gauge potential and G the gauge group. We are interested in the case when the base manifold is 4-dimensional. Then the Hodge $*$operation on $\Omega^2(M)$ satisfies $*^2=1$. If we let D_A be the covariant exterior derivative, then we have $F=D_A A = dA + 1/2[A, A]$. Then Yang-Mills equation becomes $D_A*F=0$.

In terms of component

$$\partial_\mu F_{\mu\nu}+[A_\mu, F_{\mu\nu}]=0$$

with

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu, A_\nu].$$

Note that the nonlinearity comes from the noncommutativity of the corresponding Lie algebra of the given Lie group G. If G is commutative, especially if $G=S^1$, the above equation is just Maxwell equation without source. The solutions of the Yang-Mills equation are obtained via algebraic geometry. (To be precise, the self dual solutions) This phenomena is rather surprising when you compare it with the method mentioned in section 1. In section 1, especially Yau's results are obtained after very difficult estimations. Because of the identity $*^2=1$, we can decompose $F=F^+\oplus F^-$ where $F^\pm$ are the (± 1) eigenspaces of $*$. Then we have $|F|^2=|F^+|^2+|F^-|^2$. If $F^-=0$, then we say A is selfdual. The author explained the mothod of obtaining all self-dual solutions in some place. It is still well-known unsolved problem to find non self-dual (or non anti self-dual) solutions of the Yang-Mills equations.

Now let us talk about fluid mechanics [4, 5].

From the three basic principles of physics, i. e. conservation of mass, conservation of momentum, conservation of energy, together with some reasonable assumptions, we are lead to the Euler equation for the perfect fluid, i. e.,

$$\begin{cases} \frac{\partial V_t}{\partial t}+\nabla_{v_t}v_t=-\text{grad } P_t \\ \text{div } v_t=0 \\ v_t \text{ is tangent to } \partial M \end{cases}$$

where v_t is the velocity vector field at time t on manifold M and $\nabla_{v_t}v_t$ is the covariant derivative and component-wise it is given by

$$(\nabla_{v_t}v_t)^i=\sum_j v_t\frac{\partial v_t^i}{\partial \alpha^j}+\sum_{j,k}\Gamma_{jk}^i v_t^j$$

v_t^k where Γ_{jk}^i is the christofell symbol as usual and P_t is some unknown real valued function on M called the pressure. In Euclidean space $\nabla_v v=(v\cdot\nabla)v$. The local existence and uniqueness of the solution of the Euler equation is obtained via introducing some vector field on infinite dimensional manifold. This result is a combined effort of Arnold, Ebin and Marsden. And the existence (and the uniqueness) of global solution i. e. the one defined for all t, is still open problem when dim $M\geq 3$.

When we take into consideration the fact that fluid has viscocity, we have the Navier-Stokes equation.

$$\begin{cases} \frac{\partial v}{\partial t}-\nu\nabla v+\nabla_v v=-\text{grad}P+\text{(forces)} \\ \text{div } v=0 \\ v=0 \text{ on } \partial M \end{cases}$$

The term $\nu\nabla v$ is an approximation to viscous forces. Thus the chances for a global solution are increased. There are several results in this directions due to Leray and Ladyzhenskaya. But there is no general theorem yet. These difficulties with global solutions bear on the nature of turbulence. If we are allowed to use more common language, it is still unsolved to show that "water flows" even though we get Navier-Stokes equation from the reasonable assumptions on fluid flow. Especially the existence of turbulence, whatever its defintion may be, is unsolved. Most mathematicans believe that turbulence is a result of successive losses of stability rather than non uniqueness of solution.

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