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FINITELY GENERATED CONVERGENCE SPACES

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1. Introduction

In this note we introduce the concept of finitely generated convergence spaces, and then find some internal characterizations of those spaces. Also it is shown that in the category **Cv** of convergence spaces and continuous maps, the full subcategory **FCv** of **Cv** formed by all finitely generated convergence spaces is the bicoreflective hull of the class of all finite convergence spaces and that as **Cv**, **FCv** is also cartesian closed. It is known [7] that the full subcategory **FTop** of **Top** formed by finitely generated topological spaces contains all indiscrete spaces. However we show that this is not the case for **FCv**. Finally, we give a characterization of objects of **FCv** $\cap$ **Top**.

All categorical concepts will be used in the sense of Herrlich [5, 6, 7].

2. Finitely Generated Convergence Spaces

For any set X, let $P(X)$ and $F(X)$ denote the power set of X and the set of all filters on X, respectively.

The following definition is due to Fischer [4].

DEFINITION 1. Let X be a set and $c : X \to P(F(X))$ a map. The map c is called a *convergence structure* on X if it satisfies the following:

C1) for any $x \in X$, $\dot{x} \in c(x)$.

C2) if $\mathcal{F} \in c(x)$ and $\mathcal{F} \subseteq \mathcal{G}$, then $\mathcal{G} \in c(x)$.

C3) if $\mathcal{F}, \mathcal{G} \in c(x)$, then $\mathcal{F} \cap \mathcal{G} \in c(x)$.

In this case, (X, c) is called a *convergence space.*

REMARK. In [4], convergence spaces have been called limit spaces.

NOTATION. Let (X, c) be a convergence space. If $\mathcal{F} \in c(X)$, then x is called a limit of $\mathcal{F}$, or $\mathcal{F}$ is said to converge to x, and we write $\mathcal{F} \xrightarrow{c} x$ or $\mathcal{F} \longrightarrow x$, when there is no confusion about the convergence structure c.

A filter base in a convergence space is also said to converge to x if the filter generated by the filter base converges to x.

DEFINITION 2. Let (X, c) and (Y, c') be convergence spaces and $f: X \to Y$ a map. Then f is said to be *continuous* on (X, c) to (Y, c') if for any filter $\mathfrak{F} \in c(x)$, $f(\mathfrak{F}) \in c'(f(x))$.

It is clear that the class of all convergence spaces and continuous maps between them forms a category, which will be denoted by **Cv**. Moreover, **Cv** is a properly fibred cartesian closed topological category (see [2, 6, 9]).

Now we are ready to introduce the concept of finitely generated convergence spaces.

DEFINITION 3. A convergence space (X, c) is said to be *finitely generated* if there is a final sink $(f_i : F_i \to X)_{i \in I}$ such that each F_i is a finite convergence space.

LEMMA. *Let (X, c) be a convergence space and $\mathfrak{F}$ a filter on X. Then the following are equivalent:*

a) There is a finite family $\{F_1, \cdots, F_n\}$ of finite subspaces of X and for each $k = 1, \cdots, n$, there is a filter $\mathfrak{F}_k$ on F_k converging to x in F_k, and hence in X such that $\bigcap j_{F_k}(\mathfrak{F}_k) = \bigcap \mathfrak{F}_k \subseteq \mathfrak{F}$, where j_{F_k} denotes the natural embedding of F_k into X.

b) There is a finite subspace F of X such that

$$x \in F \in \mathfrak{F} \text{ and } \mathfrak{F}|_F = \{G \mid G \subseteq F \text{ and } G \in \mathfrak{F}\} \to x.$$

c) There is a finite subspace F of X such that $F \in \mathfrak{F}$ and the principal filter $[F]$ generated by F converges to x.

Proof. a)$\Rightarrow$b). Let $F = \bigcup F_k$. Then for each k, $F \supseteq F_k$ and hence $F \in \bigcap \mathfrak{F}_k$. Thus $F \in \mathfrak{F}$, and $\bigcap \mathfrak{F}_k \subseteq \mathfrak{F}|_F$. Therefore $\mathfrak{F}|_F$ converges to x by C2 and C3.

b)$\Rightarrow$a). It is trivial.

b)$\Rightarrow$c). Since F is finite, $\bigcap \{G \mid G \subseteq F \text{ and } G \in \mathfrak{F}\} = K$ is a member of $\mathfrak{F}$ and $[K] = \mathfrak{F}|_F$ converges to x.

c)$\Rightarrow$b). Since $F \in \mathfrak{F}$, $\mathfrak{F}|_F$ contains $[F]$. Thus $\mathfrak{F}|_F \to x$.

THEOREM 1. *For a convergence space X, the following are equivalent:*

1) X is finitely generated.

2) The sink $\{j_F : F \to X \mid F$ is a finite subspace of X and j_F is the natural embedding$\}$ is final.

3) For a filter $\mathfrak{F}$ on X, $\mathfrak{F}$ converges to x iff either $\mathfrak{F} = \dot{x}$ or there is a finite subset F of X such that $F \in \mathfrak{F}$ and $[F]$ converges to x.

4) For a filter $\mathfrak{F}$ on X, $\mathfrak{F}$ converges to x iff either $\mathfrak{F} = \dot{x}$ or there is a finite subset $\{x_1, \ldots, x_n\}$ of F such that each $\dot{x}_k$ $(1 \leq k \leq n)$ converges to x and $\bigcap \dot{x}_k \subseteq \mathfrak{F}$.

Proof. 1) $\Leftrightarrow$ 2). Suppose X is finitely generated, then there is a final sink $(f_i : K_i \to X)_{i \in I}$ such that each K_i $(i \in I)$ is a finite convergence space. For each $i \in I$ let

$$K_i \xrightarrow{f_i} X = K_i \xrightarrow{h_i} f_i(K_i) \xrightarrow{j_i} X$$

be the canonical factorization, i. e., j_i is the natural embedding and $h_i(x) = f_i(x)$ $(x \in K_i)$. Since $f_i(K_i)$ is also finite, and $(j_i : f_i(K_i) \to X)_{i \in I}$ is again final, the sink $\{j_F | F$ is a finite subspace of $X\}$ is final, because it contains the sink $(j_i)_{i \in I}$.

The converse is immediate.

2) $\Leftrightarrow$ 3). By the characterization of final sinks in **Cv** (see [2, 9]), the sink $\{j_F : F \to X | F$ is a finite subspace of $X\}$ is final iff for a filter $\mathcal{F}$ on X to converge to x in X it is necessary and sufficient that either $\mathcal{F} = \dot{x}$ or $\mathcal{F}$ satisfies a) of the above lemma. Hence using the above lemma, we have the equivalence.

3) $\Leftrightarrow$ 4). It follows from the fact that for any finite subset F of X, $[F]$ coincides with $\cap \{\dot{x} | x \in F\}$.

NOTATION. The full subcategory of **Cv** formed by all finitely generated convergence spaces will be denoted by **FCv**.

THEOREM 2. *The category* **FCv** *is bicoreflective in* **Cv** *and* **FCv** *is the bicoreflective hull in* **Cv** *of the class of all finite convergence spaces.*

Proof. By the above theorem and the fact that the composition of final sinks is again final, we can conclude that **FCv** is closed under the formation of final sinks in **Cv**. Hence **FCv** is bicoreflective in **Cv**[6]. More precisely, let us find the coreflection of any convergence space (X, c). We define $c_f : X \to P(F(X))$ as follows: for any $x \in X$, $c_f(x) = \{\mathcal{F} |$ there is a finite subset $\{x_1, \ldots, x_n\}$ of X such that $\dot{x}_k$ converges to x in (X, c) $(1 \leq k \leq n)$ and $\cap \dot{x}_k \subseteq \mathcal{F}\}$. Then it is immediate that c_f is a convergence structure on X and that the identity map $1_X : (X, c_f) \to (X, c)$ is continuous. It remains to show that for any continuous map $f : (Y, c') \to (X, c)$ with $(Y, c') \in$ **FCv**, $f : (Y, c') \to (X, c_f)$ is also continuous. Suppose $\mathcal{F} \to y$ in (Y, c'), then there is a finite subset $\{y_1, \ldots, y_n\}$ such that $\dot{y}_k \to y$ in (Y, c') $(1 \leq k \leq n)$, and $\cap \dot{y}_k \subseteq \mathcal{F}$. Since $f(\dot{y}_k) = \dot{\overline{f(y_k)}} \to f(y)$ in (X, c) $(1 \leq k \leq n)$, and $f(\cap \dot{y}_k) = \cap f(\dot{y}_k) \subseteq f(\mathcal{F})$, $f(\mathcal{F})$ converges to $f(y)$ in (X, c_f). Thus $f : (Y, c') \to (X, c_f)$ is continuous.

The second part is immediate from the above theorem and the results in [6, 7].

Since **Cv** is a properly fibred topological category, the following is immediate from the above theorem.

COROLLARY. *The category* **FCv** *is a properly fibred topological category and closed under the formation of coproducts and quotients in* **Cv**.

PROPOSITION 3. *The category* **FCv** *is closed under the formation of subspaces and finite products in* **Cv**.

Proof. Let (X, c) be a finitely generated convergence space and A a subset of X. If a filter $\mathfrak{F}$ on A converges to x in the subspace (A, c_A), then $\mathfrak{F}\to x$ in (X, c). Hence there is a finite subset F of X such that $F\in\mathfrak{F}$ and $[F]\to x$ in (X, c). Since $F\subseteq A$, $F\in\mathfrak{F}$ and $[F]\to x$ in (A, c_A).

The empty product, i. e. the singleton space obviously belongs to **FCv**. Let (X, c), $(Y, c')\in$**FCv**. If a filter $\mathfrak{F}$ on $X\times Y$ converges to (x, y) in $X\times Y$, then there is a filter $\mathfrak{F}_1$ on X and a filter $\mathfrak{F}_2$ on Y such that $\mathfrak{F}_1\to x$, $\mathfrak{F}_2\to y$ and $\mathfrak{F}_1\times\mathfrak{F}_2\subseteq\mathfrak{F}$. Since (X, c) and (Y, c') belong to **FCv**, there is a finite subset F of X and a finite subset G of Y such that $F\in\mathfrak{F}_1$, $G\in\mathfrak{F}_2$, $[F]\to x$, and $[G]\to y$. Therefore $F\times G$ is finite, $F\times G\in\mathfrak{F}_1\times\mathfrak{F}_2\subseteq\mathfrak{F}$ and $[F\times G]\to(x, y)$.

Using the above proposition together with the result in [1] and the fact that **Cv** is cartesian closed, we have the following:

THEOREM 4. *The category* **FCv** *is cartesian closed.*

We note that the full subcategory **FTop** of **Top** formed by finitely generated topological spaces contains all indiscrete spaces [7].

REMARK. There is an indiscrete space which does not belong to **FCv**. For example, let (X, c) be an infinite indiscrete space and $\mathfrak{F}=\{X\}$. Then $\mathfrak{F}\to x\in X$. But there is no finite subset F of X such that $[F]\subseteq\mathfrak{F}$. Hence (X, c) is not finitely generated.

PROPOSITION 5. *A topological convergence space (X, c) belongs to* **FCv** *iff for each $x\in X$, there is a finite neighborhood F_x of x such that $\{F_x\}$ is a local base at x.*

Proof. Since (X, c) is topological, a filter $\mathfrak{F}$ on (X, c) converges to x iff it contains the neighborhood filter $\mathcal{U}_x$ of x. Hence (X, c) belongs to **FCv** iff there is a finite neighborhood F_x of x such that $[F_x]\to x$, i. e. $\mathcal{U}_x\subseteq[F_x]\subseteq\mathcal{U}_x$. Therefore (X, c) belongs to **FCv** iff $\{F_x\}$ is a local base at x and F_x is finite.

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