Bull. Korean Math. Soc.
Vol. 17, No. 2, 1981

LIE-ADMISSIBLE MUTATION ALGEBRAS

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1. Introduction

For a nonassociative algebra B, denote by B^- the algebra with multiplication $[x, y]=xy-yx$ defined on the vector space B. Then B is said to be *Lie-admissible* if B^- is a Lie algebra; that is, B^- satisfies the Jacobi identity

$$[[x, y], z]+[[y, z], x]+[[z, x], y]=0.$$

The associative algebras and Lie algebras are clearly Lie-admissible. Various types of nonassociative Lie-admissible algebras, which arise in both algebraic and physical contexts, are discussed in Myung [3] and Santilli [5].

An element x in an algebra B over a field F is said to be *flexible* if $x(yx)=(xy)x$, for all y in B, and B is said to be flexible if every element in B is flexible [1]. An element x in B is called *power-associative* if the subalgebra $F[x]$, consisting of all polynomials in x with coefficient in F, is associative and B is said to be power-associative if every element in B is power-associative [1].

In this paper we discuss the flexibility, power-associativity and some elementary properties of the mutation algebras which are derived from associative algebras.

2. Mutation of associative algebras

Let A be an associative algebra over a field F. We assume throughout the paper that the underlying base field F of A has characteristic 0 and A has an identity element 1. Let p, q be two fixed elements in A. We define the algebra $A(p, q)$, called the *(p, q)-mutation* of A, to be the algebra with new multiplication

$$x * y=xpy-yqx$$

but with the same vector space as A. Denote the associator, Lie product and Jordan product in $A(p, q)$ by $(x, y, z)^*=(x*y)*z-x*(y*z)$, $[x, y]^*=x*y-y*x$, $\{x, y\}=\frac{1}{2}(x*y+y*x)$.

Then it is easily seen that

$$[x, y]^* = x(p+q)y - y(p+q)x,$$
$$\{x, y\}^* = \frac{1}{2}[x(p-q)y + y(p-q)x],$$
$$(x, y, x)^* = x(pxq - qxp)y + y(pxq - qxp)x, \tag{2}$$

and

$$(x, x, x)^* = 2x(pxq - qxp)x. \tag{3}$$

For a fixed element $r \in A$, define $A^{(r)}$ to be the algebra with multiplication

$$x \circ_r y = xry, \quad x, y \in A,$$

but with the same vector space as A. The algebra $A^{(r)}$ is called the *r-homotope* of A and it is easily checked that $A^{(r)}$ is also associative. Thus the Lie and Jordan products in $A(p, q)$ coincide with the Lie and Jordan producte respectively in the $(p+q)$-homotope $A^{(p+q)}$ and $(p-q)$-homotope $A^{(p-q)}$. Since associative algebras are Lie-admissible and Jordan-admissible, we have that $A(p, q)$ is Lie-admissible and Jordan-admissible. However, in general, $A(p, q)$ is far from being flexible, power-associative or even third power-associative, that is $(x, x, x)^* = 0$ [2].

In [2] it is shown that $A(p, \lambda p)$, $\lambda \in F$ and $A(p, q)$ where p, q are in the center of A, are flexible and power associative.

In $A(p, 1)$, an element x in A is flexible if the associator

$$(x, y, x)^* = xpxy - x^2py + ypx^2 - yxpx$$

is zero for any $y \in A$. Thus $(x, y, x)^* = 0$ if and only if $x^2p = px^2$ and $xpx - x^2p$ is in the center of A.

If $A(p, 1)$ is flexible then for any $x \in A$, $x^2p = px^2$. Thus $(1+x)^2p = p(1+x)^2$ and hence $xp = px$. By (2) we can easily check that x is flexible in both $A(p, q)$ and $A(p, r)$ then x is flexible in $A(p, q+r)$.

We summerize these results in

THEOREM 1. *Let A be an associative algebra over F with an identity element 1. We have*

i) *For $p, q, r \in A$, assume $A(p, q)$ be flexible then $A(p, r)$ is flexible if and only if $A(p, q+r)$ is flexible,*

ii) *For $\lambda \in F$, $p \in A$, $x \in A$ is flexible in $A(p, 1+\lambda p)$ if and only if $x^2p = px^2$ and $xpx - x^2p$ is in the center of A,*

iii) For $\lambda \in F$, $p \in A$, $A(p, 1+\lambda p)$ *is flexible if and only if p is in the center of A.*

It is known that power-associative mutation algebra $A(p, q)$ need not be flexible [4], and in general flexible Lie-admissible algebra need not be power-associative. However in $A(p, q)$ we have

THEOREM 2. *Let A be an associative algebra over F with an identity element* 1. *If an element x in A is third power-associative in $A(p,q)$, that is $(x,x,x)^*=0$, then x is power-associative in $A(p,q)$. In particular if $A(p,q)$ is flexible, then $A(p,q)$ is power-associative.*

Proof. By (3), $(x,x,x)^*=0$ means

$$xpxqx=xqxpx. \tag{4}$$

Hence it is easy to check that for any natural number n,

$$\begin{aligned}[x(p-q)]^n xpx &= xp[x(p-q)]^n x,\\ [x(p-q)]^n xqx &= xq[x(p-q)]^n x.\end{aligned} \tag{5}$$

Let $x^{*n}=x^{*n-1}*x$, $n>1$, and denote $x^1=x$ and $x^0=1$ in A. It is sufficient to prove that for any $n\geq 2$, and natural numbers i,j, with $i+j=n$,

$$x^{*i}*x^{*j}=[x(p-q)]^{n-1}x. \tag{6}$$

For $n=2$, it is obvious and for $n=3$, from hypothesis $x^{*2}*x=x*x^{*2}$, and by (5)

$$\begin{aligned}x^{*2}*x&=[x(p-q)]xpx-xq[x(p-q)]x\\ &=[x(p-q)]xpx-[x(p-q)]xqx=[x(p-q)]^2x.\end{aligned}$$

Assume (6) for $i+j<n$, then for $i+j=n$,

$$\begin{aligned}&x^{*i}*x^{*j}\\ &=[x(p-q)]^{i-1}xp[x(p-q)]^{j-1}x-[x(p-q)]^{j-1}xq[x(p-q)]^{i-1}x\\ &=[x(p-q)]^{n-2}xpx-[x(p-q)]^{n-2}xqx=[x(p-q)]^{n-1}x,\end{aligned}$$

and completes the proof.

It can be shown that $A(p,1)$ is power-associative if and only if p is in the center of A, hence we have

COROLLARY. *Let A be an associative algebra over F, and $\lambda\in F$, then $A(p, 1+\lambda p)$ is power-associative if and only if p is in the center of A.*

In [4] Oehmke discussed the flexibility and powerassociativity in $A(p, 1-p)$.

3. Ideals

For a nonassociative algebra B, I is an ideal of B if I is a subspace of B and for every $x\in I$, $y\in B$, xy and yx are in I. Denote A^{+} the algebra with multiplication $\{x,y\}=\frac{1}{2}(xy+yx)$ defined on the vector space B.

Let A be an associative algebra over F, and I be an ideal of $A(p,q)$. Assume $pq=qp$. Then for any $x\in A$ and $y\in I$, $x*y=xpy-yqx\in I$ and

$y*x=ypx-xqy\in I$. Hence

$$x(p-q)y+y(p-q)x\in I,$$
$$x(p+q)y-y(p+q)x\in I. \quad (7)$$

Thus I is an ideal of $(A^{(p+q)})^-$ and an ideal of $(A^{(p-q)})^+$. Setting $x=p+q$, $x=p-q$ respectively in (7), we have

$$(p-q)(p+q)y\in I,$$
$$y(p+q)(p-q)\in I. \quad (8)$$

for any $y\in I$. Denote the multiplication of the associative algebra $A^{(p+q)}$ by $\circ$, that is, $x\circ y=x(p+q)y$. We assume also $x\circ(p-q)\circ y+y\circ(p-q)\circ x\in I$ for any $x\in A$, $y\in I$. Then by (7) and (8), the elements

$$x\circ(p-q)\circ y-y\circ x\circ(p-q),$$
$$y\circ x\circ(p-q)-x\circ y\circ(p-q),$$
$$x\circ y\circ(p-q)-y\circ(p-q)\circ x, \text{ and}$$
$$y\circ(p-q)\circ x+x\circ(p-q)\circ y$$

are in I. Adding these elements, we have

$$x\circ(p-q)\circ y\in I \text{ and } y\circ(p-q)\circ x\in I \quad (9)$$

THEOREM 3. *Let A be the algebra of all $n\times n$ matrices over a field F with characteristic 0. If $p+q$ is a invertible element of A and $p\neq\pm q$, then $A(p,q)$ is simple.*

Proof. It can be checked that the map $x\to x(p+q)^{-1}$ is an isomorphism of $A(p,q)$ onto $A((p+q)^{-1}p,\ (p+q)^{-1}q)$. Since $(p+q)^{-1}p$ commutes with $(p+q)^{-1}q$, we may assume $p+q=1$ and $pq=qp$. Let I be a proper ideal of $A(p,q)$. Since $A^{(p+q)}=A$, from (7), (8), (9) we have I is an ideal of A^- and the elements $(p-q)y$, $y(p-q)$, $x(p-q)y$ and $y(p-q)x$ are in I for any $x\in A$, $y\in I$. Since the Lie algebra $sl(n,F)$ is simple for $n\geq 2$, I must be the set of trace 0 matrices S or the center Z of A.

If $I=S$, then $(p-q)$ is a scalar matrix because $(p-q)y\in S$ and $y(p-q)\in S$ for any $y\in S$. Let x be an element of A. Since $(p-q)\neq 0$,

$$[x(p-q)^{-1}](p-q)y=xy\in I$$

and

$$y(p-q)[(p-q)^{-1}x]=yx\in I.$$

Hence I is an ideal of A which is impossible.

If $I=Z$ then $(p-q)\in Z$ because Z is the set of all scalar-matrices. Thus for $x\in A$, $l\in Z$, $x(p-q)l=\lambda x\in Z$ for some $\lambda\in F$, and contradiction.

If $n=1$, then A is isomorphic to F and hence $(p-q)$ is invertible. Thus we can easily show that $A(p,q)$ is a simple algebra.

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