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# ON STABLE MINIMAL SURFACES IN $R^4$.

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## 1. Introduction.

It is very fascinating to observe that the graph $(z, f(z))$ of any holomorphic function $f: \mathbf{C} \to \mathbf{C}$ is a minimal surface in $R^4 = \mathbf{C} \times \mathbf{C}$. Moreover the Wirtnger's inequality tells us that these are stable. And recently it is shown by D. Fischer-Collbrie and Schoen, independently by De Carmo and Peng that plane is the only stable minimal surface in $R^3$. Abundence of stable minimal surfaces in $R^4$ makes the following conjecture; If $M$ is a stable mimal surface in $R^4$, then there exists a complex structure $J$ and a decomposition $R^4 = \mathbf{C} \times \mathbf{C}$ such that $M$ is a graph of a holomorphic function. The purpose of this article is to give a small contribution to this conjecture.

## 2. Theorems

We start with giving some basic theorems and defining notations to be used. Basic reference is Lawson's [3].

Let $\overline{M}$ be a riemannian $\bar{m}$-manifold and $M \to \overline{M}$ a topologically embedded submanifold of dimension $m$. Denote the metric on $\overline{M}$ by $\langle \cdot, \cdot \rangle$ and the associated riemannian connection by $\bar{\nabla}$. Then we can decompose any vector $X \in T_p(\overline{M})$ as $X = (X)^T + (X)^N$, where we have an orthogonal splitting

$$T_p(\overline{M}) = T_p(M) \oplus N_p(M).$$

The induced metric on $M$ give a connection via

$$X^Y = (\bar{\nabla}_X Y)^T.$$

Then the second fundamental form $B_{X,Y}$ at $P$, where $X, Y \in T_pM$, is defined as

$$B_{X,Y} = (\bar{\nabla}_X Y)^N$$

Then $B_{X,Y} = B_{Y,X}$, i.e., $B_p$ represents a symmetric bilinear map of $T^p(M)$ into $N_p(M)$.

Thus we can define $K_p = \text{trace}\ (B_p)$ at each $p$. $K$ is a smooth field of normal vectors on $M$ called the mean curvature vector field. Locally, if $\varepsilon_1, \cdots\cdots, \varepsilon_m \in T_p$ are pointwise orthonormal fields, then

$$K_p = \sum_{k=1}^{m} (\bar{\nabla}_{\varepsilon_k} \varepsilon_k)^N.$$

DEFINITION: $M \to \bar{M}$ is called a minimal submanifold if $K_p = 0 \ \forall p \in M$.

This definition of a minimal submanifold satisfies our intuition because we have the following variational formula.

THEOREM 1 [3]. *Let $f(t) : M \to \bar{M}$ be one parameter of imbedding of $M$ into $\bar{M}$ and $A(t)$ the volume of $M$ with induced metric. Then*

$$\left.\frac{dA}{dt}\right|_{t=0} = -\int_M \langle K, E \rangle d V_0.$$

*Here $K$ is mean curvature of $A(0) : M \to \bar{M}$ and $E = F_* \left.\frac{\partial}{\partial t}\right|_{t=0}$, where $F : I \times M \to M$ defined as $F(t, x) = f_t(x)$.*

*Proof* : Refer to Lawson's [3].

As usual in critical point theory, it is natural to ask wheather this extremum is local minimum or local maximum. When it is local minimum, it is called a stable minimal submanifold. To see if it is stable or not, we must compute second variation. Before giving the formula, let us set notations. For any $X \in X(M)$ and normal vector field $\nabla_X \nu$ by the formula $\nabla_X \nu = (\bar{\nabla}_X \nu)^N$ Then the resulting map $\nabla_X : \Gamma(N(M)) \to \Gamma(N(M))$ is a connection on the normal bundle of $M$. Using this connection we define the laplacion $\nabla$ : $\Gamma(N(M)) \to \Gamma(N(M))$ by setting, at each point $\in M$,

$$\nabla \nu(p) = \sum_{j=1}^{m} (\nabla_{\varepsilon_j} \nabla_{\varepsilon_j} \nu - \nabla_{\nabla \varepsilon_j \varepsilon_j} \nu)(p),$$

where $\varepsilon_1, \cdots\cdots, \varepsilon_m$ $\bar{M}_p$ are local pointwise orthonormal tangent vectors. Recall that for vectors $X, y \in T_p(\bar{M})$ we define a curvature transformation

$$\bar{R}_{X,Y} : T_p(\bar{M}) \to T_p(\bar{M})$$

by

$$\bar{R}_{\tilde{X},\tilde{Y}} Z = (\bar{\nabla}_X \bar{\nabla}_Y \tilde{Z} - \bar{\nabla}_{\tilde{Y}} \bar{\nabla}_{\tilde{X}} \tilde{Z} - \bar{\nabla}_{[\tilde{X},\tilde{Y}]} \tilde{Z})_p$$

A similar transformation $R_{X,Y} : T_p(M) \to T_p(M)$ is defined for the submanifold $M$. Then the two arerelated by the famous Gauss curvature formula

$$\langle \bar{R}_{X,Y} Z, W \rangle - \langle R_{X,Y} Z, W \rangle = \langle B_{X,Z}, B_{Y,W} \rangle - \langle B_{X,W}, B_{Y,Z} \rangle$$

for all $X, Y, Z, W \in T_p(M)$.

At each point $p\in M$, we define a linear transformation $\tilde{R}: N_p(M)\rightarrow N_p(M)$ by $\tilde{R}(\nu)=\sum_{i=1}^{m}(\bar{R}_{e_j}, \nu e_j)^N$. Also noting that $B_p$, the second fundamental form at $p$, can be thought as an element in $\mathrm{Hom}(ST_p(M), N_p(M))$ where $ST_p(M)$ is the set of symmetric endomorphisms of $T_pM$. Then we define $B_p: N_p(M)\rightarrow N_p(M)$ by $B=B\circ {}^tB$. Note that for $\mu, \nu\in N_p(M)$, $<B(\nu), \mu>=<B\circ {}^tB(\nu), \mu>=<{}^tB(\nu), {}^tB(\mu)>=\sum<\nu, B_{e_i, e_j}><\mu, B_{e_i, e_j}>$ where $\{e_1, \cdots\cdots, e_m\}$ is any orthonormal basis of $T_pM$.

Now we have the second variational formular.

THEOREM 2. *Notations as above, we have*

$$\left.\frac{d^2A}{dt^2}\right|_{t=0}=\int_M<-\Delta E+\bar{R}(E)-B(E), E>dV.$$

*Proof*: Refer to Lawson's [3].

Theorem 2 has several consequences. First note that the symmetric differential operator $L=-\Delta+\bar{R}-B$ defined in $\Gamma_0(N(M))$ is elliptic. From the general theory of such operators we know that it can be diagonalized on $\Gamma_0(N(M))$ with eigenvalues $\lambda_1<\lambda_2<\cdots\rightarrow\infty$ where each eigenspace $V_{\lambda_j}$ is finite dimensional.

Note that the stability of $M$, i.e., $\frac{d^2A}{dt^2}\geq 0$ is equivalent to

$$\int_M|\nabla E|^2+\bar{R}(E)\cdot E-|{}^tBE|^2dV\geq 0$$

for any $E\in\Gamma_0(N(M))$.

From now on we restrict ourselves to the case when $\bar{M}=R^4$. Then $\tilde{R}(E)=0$ for all $E\in\Gamma_0(N(M))$. In this special case $L=-\Delta-B$, and $\frac{d^2A(t)}{dt^2}\geq 0$ is equivalent to $\int_M|\nabla E|^2-|{}^tB(E)|^2dV\geq 0$.

Before giving our theorem, we give a general fact on reduction of codimension of isometric immersions, which is due to Erbacher [2]. The first normal space $N_1(p)$ at $p\in M^n$ is defined to be the orthogonal complement in $(M_p^n)$ of $\{\xi(p)\in(M_p^n): A_{\xi(p)}\equiv 0\}=N'$. Here $A_{\xi(p)}: T_pM\rightarrow T_pM$ defined as $<A_{\xi(p)}X, Y>=<B_p(X, Y), \xi_p>$ in $T_p\bar{M}$.

THEOREM 3. *If $M\rightarrow\bar{M}$ is an isometric immersion and $\bigcup_{p\in M}N_1(p)$ form a subbundle of the normal bundle $N(M)$ and more over it is parallel,* i.e., $\nabla_Z\xi\in N_1$ *for all $\xi\in N_1$ and $Z\in T_pM$, then there exists a totally geodesic submanifold $M'$ of $\bar{M}$ such that* $\dim M'=n+k$ *and the image of $M$ lies in $M'$, where* *k is the dimesion of the bundle* $N_1$.

Now we give our theorem.

THEOREM 4. *Let* $M \to R^4$ *be a minimal immersion and assume the first normal space* $N_1(p)$ *form a line bundle over M and the normal curvature of the immersion, defined as* $K_N = \langle R^2{}_{XY}\xi, \eta \rangle$ *where* $X, Y, \xi, \eta$ *are orthonormal set of* $T_pM$, $(M_p)^\perp$ *respectively, vanishes. Then our minimal immersion is in* $\mathbf{R}^3$. *Hence if it is stable then it must be a plane.*

*Proof* : Due to the Erbacher's theorem and the result of De Carmo, Peng, Fisher-Collbrie and Schoen, all we have to do is to show the first normal spaces form a parallel subbundle of the normal bundle, or equivalently, its complement $N'$ is parallel because our connection $\nabla$ is compatible with the metric on $N(M)$. That is, it suffices to show $\nabla_{X_i}\xi \in N'$ if $\xi \in N'$. From the definiti on of $N'$ we have to show $A_{\nabla X_i}\xi \equiv 0$ if $\xi \in N'$. Where $X_i$ are coordinate vector fields in a neighborhood of $p$. Because the ambient space is $\mathbf{R}^4$, $\bar{R}_{X_iX_j}\xi_\alpha = 0$. But

$$\begin{aligned}\bar{R}_{X_iX_j}\xi_\alpha &= \bar{\nabla}_{X_j}\bar{\nabla}_{X_i}\xi_\alpha - \bar{\nabla}_{X_i}\bar{\nabla}_{X_j}\xi_\alpha \\ &= \bar{\nabla}_{X_j}(\nabla_{X_i}\xi_\alpha - A_{\xi_\alpha}X_i) - \bar{\nabla}_{X_i}(\nabla_{X_j}\xi_\alpha - A_{\xi_\alpha}X_j).\end{aligned}$$

Using the fact that $\bar{\nabla}_{X_i} = -A_{\xi_\alpha}X + \nabla_{X_i}\xi_\alpha$ again, we have $R^\perp{}_{X_jX_i}\xi_\alpha - A_{\nabla X_i\xi_\alpha}X_j + A_{\nabla X_j\xi_\alpha}X_i = 0$. And from the assumption that $K_N \equiv 0$, we have

$$A_{\nabla X_i\xi_\alpha}X_i - A_{\nabla X_j\xi_\alpha}X_j = 0$$

Note that

$$\langle R^\perp{}_{X_iX_j}\xi, \eta \rangle = \langle [A_\xi, A_\eta]X_i, X_j \rangle.$$

All of $A_\xi$ can be diagonalized simultaneously since $K_N \equiv 0$.

Hence $A_{\nabla X_j\xi_\alpha}X_i = a_j{}^iX_i$ Therefore, the above equality gives $A_{\nabla X_j\xi_\alpha}X_j = 0$ if $i \neq j$. Since the immersion is minimal we have $\sum_{i=1}^{n} \langle A_{\nabla X_j\xi_\alpha}X_i, X_i \rangle = 0$ Since only non zero contribution comes when $i = j$. We have $A_{\nabla X_j\xi_\alpha}X_j = 0$, too. That is, $A_{\nabla X_j\xi_\alpha}X_i = 0$ for every $i, j$. Hence $\nabla_{X_j}\xi_\alpha \in N'$. q. e. d.

REMARK. The above argument is essentially due to Colares and do Carmo. It would be very interesting to find more general condition than $K_N \equiv 0$.

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