Bull. Korean Math. Soc.
Vol. 17, No. 2, 1981

THE PROPER EMBEDDINGS OF LORENTZIAN MANIFOLDS INTO A EUCLIDEAN (PSEUDO-EUCLIDEAN) SPACE BY SOLUTIONS OF THE D'ALEMBERTIAN EQUATIONS

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1. Introduction

In [5], I proved that any connected open Lorentzian manifold M can be topologically embedded into a euclidean (pseudo-euclidean) space by solutions of the d'Alembertian equation. This realizations of an abstract manifold is still not quite beautiful in some sence. For example, the figure 6 is topological embedding of a line into 2-dimensional euclidean space. However, it is not proper embedding of the line because of limit point of the line on one side. Thus it is desirable to get the proper embedding of our M for the purpose of applying the results to Mathematics and Physics.

In this paper we will show that any globally hyperbolic connected open Lorentzian manifold can be properly embedded into a euclidean (pseudo-euclidean) space by solutions of the d'Alembertian equation on M.

2. Preliminaries

Throughout this paper a manifold M will be restricted to an orientable space which is C^∞, connected, non-compact, paracompact, Hausdorff, time orientable, and has dimension m. A Lorentzian metric on M means the mo dules of the signature of the metric is $m-2$ or $2-m$. This manifold, M, with a Lorentzian metric is called a Lorentzian manifold. By taking the covariant and contravariant derivatives of a scalar field the divergence of the gradient of a scalar field, becomes an invariant second order linear partial differential operator,

$$\nabla_i \nabla^i u = \|g\|^{-\frac{1}{2}} \frac{\partial}{\partial x_i}\left(\|g\|^{\frac{1}{2}} g^{ij} \frac{\partial u}{\partial x_j}\right),$$

where g is a Lorentzian metric, $|g|$ determinant of g, $\|g\|$ absolute value of g, that is,

$$\|g\|^{\frac{1}{2}} = \sqrt{|\det g_{ij}|},$$

This work is partially supported by the grant of the Ministry of Education, Republic of Korea.

and g^{ij} is a component of the inverse of the matrix (g_{ij}) consisting of the metric tensor. This differential operator is called the d'Alembertian and will be denoted by $\Box$, and we will call the d'Alembertian equation $\Box u=0$. $\Box$ is strong, self-adjoint, continuous, hyperbolic, second order, linear partial differential operator. Moreover, it is invariant by Lorentzian transformations, and the d'Alembertian equation is a kind of the wave equations. The vector wave equations are also treated for our purpose.

The future dependence domain $D^+(q)$ is defined by the set of all points p in a geodesically convex demain Ω being time orientable that can be reached along future directed time-like geodesics from q. And the past dependence domain $D^+(q)$ is also defined similarly. The future emission $J^+(q)$ at q is the set comprising q together with all points from q by a positively oriented non-space-like curve, that is, one whose tangent vector has non-positive length-squared. And the past emission $J^+(q)$ at q is also defined similarly. We know that

$$\overline{D^+(q)}=J^+(q),\ \ \overline{D^-(q)}=J^-(q),\ \ J^+(A)=\bigcup_{q\in A}J^+(q),\ \ and\ \ J^-(A)=\bigcup_{q\in A}J^-(q)$$

for $q\in\Omega$ and a subset A, where $^-$ indecates closure.

A connected open set Ω will be called a causal domain if

1) there is a geodesically convex domain Ω_0 such that $\Omega\subset\Omega_0$

and

2) $J^+(q)\cap J^-(q)$ is a compact subset of Ω, or void for all pairs of points p, q in Ω.

A lorentzian manifold is said to be globally hyperbolic if the strong causality assumptions (see [5]) hold and if for any two pair p, q in M, $J^+(q)\cap J^-(p)$ is compact.

Let $\mathbf{R}_q{}^p$ be apseudo euclidean space, where p and q are non-negative integers. If $q=0$, it is a euclidean space. A mapping $u: M\to\mathbf{R}_q{}^p$ will be said a topological C^k-embedding if

1) u is of differentialibility class C^k $(k\geq 0)$ and its differential mapping u_* has rank m at all points in M

2) u is injective.

Let $L(f)$ be the limit set of f, that is,

$L(f)=\{y\in\mathbf{R}_q{}^p|$ for each divergent sequence $\{x_n\}$ in M, $f(x_n)$ converges to y in $\mathbf{R}_q{}^p\}$.

In addition to (1) and (2) above, if $L(u)$ has a null intersection with $u(M)$, then u is called a proper embedding. The following statements for the proper mapping of M are equivalent:

3) a mapping $f: M\to\mathbf{R}^p$ is proper if and only if for each compact subset

K of $\mathbf{R}^p$, $f^{-1}(K)$ is compact

4) let f be an embedding of M into $\mathbf{R}^p$. f is proper if and only if f is a closed mapping

5) let M^* and $(\mathbf{R}^p)^*$ be one-point compactification of M and $\mathbf{R}^p$ respectively, and extand $f: M \to \mathbf{R}^p$ to a mapping $F: M^* \to (\mathbf{R}^p)^*$ by letting $F(\infty)=\infty$. Then f is proper continuous if and only if F is continuous. Let $\mathcal{D}'(M)$ be a set of all distributions on M. Then the strong topology of distributions on $\mathcal{D}'(M)$ is defined by taking basic neighborhoods of zero in $\mathcal{D}'(M)$ as sets

$$V(A, \varepsilon)=\{u \in \mathcal{D}'(M) \mid |\langle u, \phi\rangle|<\varepsilon \text{ for all } \phi \in A\}$$

for an arbitrary positive ε and an arbitrary bounded subset A of the space of test functions $C_0(M)$. This topology is equivalently defined by the semi-norms (see [4]). If a manifold M satisfies the global hyperbolicity M admits its global time functions and the cauchy surfaces (say cauchy surfaces). Moreover, in this case the following facts are true (Hawking and Ellis [2], Friedlander [1]);

6) $M \cong \mathbf{R} \times N$, where N is $(m-1)$-dimensional manifold and cauchy surface

7) $M=D^+(N) \cap D^-(N)$.

In [5], the following results were shown:

LEMMA *. *If X is acompact subset of a causal domain Ω_0, the $\mathcal{E}(X)$ is a dense subset of $\mathcal{B}(\Omega_0)$ in the strong topology of the space of distributions, where $\mathcal{E}(X)$ is a set of all embeddings of X to $\mathbf{R}^{2m+1}$ by solutions of the d'Alembertian equation, $\mathcal{B}(\Omega_0)$ is a subspace of $\mathcal{D}'(\Omega_0)$ consisting of all solutions of the d'Alembertian equation on Ω_0, and $\mathcal{D}'(\Omega_0)$ is $2m+1$-valued distribution space on Ω_0.*

THEOREM *. *If M is a globally hyperbolic connected open Lorentzian manifold of dimension m, then M can be topologically embedded into $\mathbf{R}^{2m+1}$ by the mapping whose components are solutions of the d'Alembertian equation.*

THEOREM **. *If M is a globally hyperbolic connected open Lorentzian manifold with C^k-Lorentzian metrics $(k \geq 3)$ and dimension m, them M can be topologically embedded into $\mathbf{R}_1^{p+1}$ $(p \geq 2m+1)$ by the mapping whose components are solutions of the d'Alembertian equations. Moreover the induced metric from this mapping is Lorentzian.*

REMARKS. Non-compactness of our manifold, that is, open manifold, is necessary for our embedding problem. Suppose the manifold admits compactness. Then this allows violations of the physical world and mathematically presents many obstacles for the d'Alembertian equation. And it happens troubles for the Lorentzian metrics on our manifold (see Friedlander [1], Hawking and Ellis [2], Hörmander [3] and Leray [6]). Since the basic tools of our embedding problems are solutions of the d'Alembertian equation on M, we are concerned with the causal domains for the well-posed problems of the d'Alembertian equation (see Friedlander [1], and Leray [6]).

3. The main Results

Before we prove the main theorem, the lemmas will be constructed. The following lemma is based on the techniques of Friedlander [1] and Hawking and Ellis [2].

LEMMA 1. *Let M be a glob ally hyperbolic Lorentzian Manifold and $\{a_i\}$ be a sequence of real numbers. Then there exist a solution u of the d'Alembertian equation on M, and a s∂quence of compact subsets $\{X_i\}$ of M such that*

$$u : M \to \mathbf{R} \text{ and } u \geq a_i \text{ on } X_i,$$

where $X_i \subset X_{i+1}$ and $\bigcup_i X_i = M$.

Proof. Let N be a cauchy surface of the globally hyperbolic Lorentzian Manifold M. Then $M = D^+(N) \cup D^-(N)$ and the fundamental solutions of the d'Alembertian equation on M vanishies on $M - J^+(N)$ or $M - J^-(N)$. It is enough to consider our desired solutions of the equation on $J^+(N)$ because solutions on $J^-(N)$ can be similarly taken, and the sum of these two solutions can be a global solution of the equation on M which satisfies the lemma.

Since $M \cong \mathbf{R} \times N$, each $\{b\} \times N$ is a cauchy surface. In case of the space-time that is, the 4-dimensional Lorentzian manifold which is Einstein general relativity space, we can choose $\mathbf{R}$ as a global time on M.
Thus we can first consider a coordinate neighborhood U of a point of N such that

$$U \cap N = \{x \mid x_1 = 0\}.$$

U may be taken as a causal neighborhood of the point of N if necessary. By letting $x = (x_1, x^*)$, the cauchy data given

$$u(0, x^*) = u_0(x^*) \geq a \text{ for a given real number } a,$$

and

$$\partial_1 u(0, x^*) = u_1(x^*) \geq 0.$$

Let us assume these functions, u_0 and u_1, are smooth on $U \cap N$. $u_n(x^*) = \partial_1{}^n u(0, x^*)$ can be computed, for $n \geq 2$, from u_0, u_1, and the equation $\partial_1{}^{n-2}$

$(\Box u)=0$. To convert the formal series in x_1 nto a convergent series, let $\sigma(t)\in C_0^\infty(\mathbf{R})$ be such that

$$0\leq\sigma\leq 1,$$
$$\sigma=1 \text{ for } |t|\leq\frac{1}{2},$$
$$\sigma=0 \text{ for } |t|\geq 1.$$

Then, choosing a sequence of positive number $\{c_n\}$, which increases sufficiently rapidly, the series

$$\sum_{n=0}^{\infty} u_n(x^*)\frac{x_1{}^n}{n!}\sigma(c_nx_1)$$

converges in U to a function v, and uniformly in the maximal compact subset of U (see Friedlander [1]). By our assumptions above and §4−§6 in Friedlander [1], v is a formal solution of the d'Alembertianequation on U with the cauchy data on $U\cap N$. Moreover, $v\geq a$ on the maximal compact subset of U. It is possibly guaranteed by the condition, $u_1(x^*)\geq 0$ (see the minimum principle in protter and weinberger [7], *e.g.*, assigning more restricted data as $u_1(x^*)=0$ and $u_n(x^*)\leq 0$ ($n\geq 3$ odd) if x_1 is negative, the series above has values which are larger than a on the maximal compact subset of U.

Now let $\{U_i\}$ be a locally finite covering of N by coordinate neighborhoods of this type, and denote the union of the U_i by W. Let $\{\phi_i\}$ be a partition of unity defined on W, subordinated to this covering. By carrying out the constructions just described in each v_i with the cauchy data on $U_i\cap N$, and letting $v=\sum_i v_i\phi_i$, v is a solution of the equation on W with the cauchy data on $W\cap N$ (see more detailed explanations in Friedlander [1]) such that it vanishes outside of W because we have, in terms of the local coordinate in each U_i,

$$\partial_1 v=\sum_i \phi_i\partial_1 v_i+\sum_i v_i\partial_1\phi_i,$$
$$\sum_j \phi_j=1,\ \sum_j \partial_1\phi_j=0.$$

Moreover, since for each maximal compact subset of U_k and given real number a_k we have

$$v=\sum_i v_i\phi_i \geq a_k\sum_i \phi_i=a_k$$

by the construction of U_k, $v\geq a_k$ on the maximal compact subset of U_k.

Let X be the characteristic function of $J^+(N)$ such that

$$X=1 \quad \text{on } W,$$
$$=0 \quad \text{on } J^+(N)-W.$$

Consider the equation

$$\Box\omega = X \cdot \Box v \text{ on } J^+(N).$$

Then,

$$0 = \Box\omega = X \cdot \Box v = 1 \cdot \Box v = \Box v \text{ on } W$$

could be $\omega = v$ on W. Therefore we can similarly solve the equation

$$\Box\omega = X \cdot \Box v = 0 \text{ on } J^+(N),$$

i. e.,

$$\Box\omega = X \cdot \Box v = 0 \cdot \Box v = 0 \text{ on } J^+(N) - W.$$

In other words, it would be similarly done by taking the cauchy data on the cauchy surface $\{b\} \times N$ for some b in $\mathbf{R}$. This solution ω may possibly be taken a non-negative function by assigning suitable cauchy data as above. Thus $u = v + \omega$ is a solution of the d'Alembertian equation on $J^+(N)$. Let $\{X_i\}$ be a sequence of compact subsets such that

$$X_i \subset X_{i+1}, \quad \cup_i X_i = J^+(N),$$

and X_i contains the maximal compact subset of U_i for all $j \leq i$, but not the maximal compact subset of U_k for $k > i$. Then, by our constructions

$$u \geq a_i \text{ on } X_i \text{ for each } i \text{ and given real number } a_i.$$

The proof of our lemma is complete.

We will show examples which are related to Lemma 1 by taking Minkowski space. Let M be $\mathbf{R}^m$ with Lorentzian metric. We know that if the cauchy problem is

$$\Box u = 0 \text{ on } \mathbf{R}^m,$$
$$u(0, x) = u_0(x),$$
$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = u_1(x),$$

and u_0, u_1 are smooth functions on $\mathbf{R}^{m-1}$, then a representation of global solution of the d'Alembertian equation on $\mathbf{R}^m$ is

$$u(t, x) = u_0(x)\,(x^*)\frac{\partial}{\partial t}(G_+ - G_-) + u_1(x)\,(x^*)\frac{\partial}{\partial t}(G_+ - G_-),$$

where G_+ and G_- are fundamental solutions of the equations for the past and future emissions respectively, and (x^*) means convolution with respect to x (see Tréves [8]). From this representation, let us compute the next examples.

EXAMPLES. Consider the cauchy problem;

$$\Box u=0 \text{ on } \mathbf{R}^2,$$
$$u(0, x)=u_0(x),$$
$$\left.\frac{\partial u}{\partial t}\right|_{t-0}=u_1(x),$$

where u_0 and u_1 are C^k or C^∞ on $\mathbf{R}$, and in addition, for given real number a_i, $u_0 \geq a_i$ and $u_1 \geq 0$ on the cauchy surface in Δ^i, where Δ^i is past or future emission. Then, by usual computations,

$$u(t, x)=\frac{1}{2}\{u_0(t+x)+u_0(t-x)+\int_{x-t}^{x+t} u_1(y)\,dy\}$$
$$\geq \frac{1}{2}\{2a_i+b\} \geq a_i \quad \text{on } X_i,$$

where X_i is compact subset of Δ^i containing the cauchy surface, and

$$b=\int_{x-t}^{x+t} u_1(y)\,dy \text{ on } x_i$$

Next, consider the cauchy problem;

$$\Box u=0 \text{ on } \mathbf{R}^3,$$
$$u(0, x, y)=u_0(x, y),$$
$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=u_1(x, y),$$

where u_0 and u_1 are C^k or C^∞ on $\mathbf{R}^2$, and in addition, for given a_i,

$$u_0(x, y) \geq a_i, \quad u_1(x, y) \text{ and } u_{tt}|_{t=0} \geq 0$$

on the cauchy surface in Δ^i.
Then we have the solution

$$u(t, x, y)=u_0(x, y)+\frac{1}{2\pi}\iint_{w^2+z^2<t^2} \frac{u_1(x+w, y+z)}{\sqrt{t^2-w^2-z^2}}\,dw\,dz$$
$$+\frac{1}{2\pi}\int_0^t dt \iint_{w^2+z^2<t^2} \frac{u_{tt}(x+w, y+z)}{\sqrt{t^2-w^2-z^2}}\,dw\,dz.$$

Moreover,

$$u(t, x, y) \geq a_i+b \geq a_i \text{ on } x_i$$

where

$$b=\frac{1}{2\pi}\iint_{w^2+z^2<t^2} \frac{u_1(x+z, y+z)}{\sqrt{t^2-w^2-z^2}}\,dw\,dz$$
$$+\frac{1}{2\pi}\int_0^t dt \iint_{w^2 z^2<t^2} \frac{u_{tt}(x+w, y+z)}{\sqrt{t^2-w^2-z^2}}\,dw dz \quad \text{on } X_i.$$

Similarly we can get that, for given a_i, $u(t, x, y, z) \geq a_i$ on suitable compact subset x_i of $\mathbf{R}^4$ and the m-dimensional case can also be done by the similar way. We took the above representations of solutions of the equations from Tréves [8] and Protter and Weinberger [7]. Moreover the additional condition of the cauchy data above are closely related to the minimum principle, that is, if

$$\frac{\partial^{n-1}}{\partial t^{n-1}}(\Box u) \leq 0 \text{ in } \Omega, \text{ and}$$

$$\frac{\partial^i u}{\partial t^i}(0, x_1, \cdots, x_n) \geq 0, \quad i=1, \cdots, n,$$

where Ω is a domain of $t \geq 0$ containing the characteristic cone of each of its points, then the minimum value of u in Ω must occur at $t=0$ (see Protter and Weinberger [7]). Therefore our examples described above are reasonable and, as we know, the Huygen's principle does not interfere with our example.

Now we will show the existence of proper mapping of M to $\mathbf{R}^q$ whose components are solutions of the d'Alembertian equations.

LEMMA 2. *If M is globally hyperbolic, then there exists a proper mapping $u : M \to \mathbf{R}^{m+1}$ whose components are solutions of the d'Alembertian equations on M.*

Proof. By Lemma 1, for given real numbers $\{a_j\}$, there exist solutions u_i, $i=1, 2, \cdots, m+1$, of the equations on M such that if $\{X_j^i\}$ is one of Lemma 1, then

$$u_i(p) \geq a_j \text{ for each } i \text{ and all } p \in x_j^i.$$

By assigning suitable a_j,

$$\lim_{j \to \infty} \inf_{p \in} X_j^i \ u_i(p) = \infty \text{ for each } i.$$

Therefore the mapping $u : M \to \mathbf{R}^{m+1}$ defined by

$$p \to (u_1(p), \cdots, u_{m+1}(p))$$

is proper by (5) on page 3.
The proof of the lemma is complete.

Now, using these lemmas and the results in the previous chapter, the following theorem can be proved.

THEOREM. *Any connected open Lorentzian manifold M of dimension m which is global hyperbolic can be properly embedded into $\mathbf{R}^{2m+1}$ by solutions of the d'Alembertian equations.*

proof. By Theorem* there exists an topological embedding

$$u=(u_1, \cdots, u_{2m+1}) : M \to \mathbf{R}^{2m+1}$$

whose components are solutions of the d'Ale mbertian equations on M.

Let a sequence $\{a_j^i\}$ of real numbers by

$$a_j^i = \max(2 \max_{p \in X^i_j} (\max_{1 \le k \le 2m+1} |u_k(p)|), \ j),$$

where $\{X_j^i\}$ is one of Lemma 1,

$$i=1, \cdots, m+1, \text{ and } j=1, 2, \cdots,$$

Then $a_j^i \ge j$ for each i.

Now, we can choose solutions v_i of the equations on M, as those of Lemma 1 on x_j^i for each i, such that, by Lemma2, the mapping

$$v=(v_1, \cdots, v_{m+1}) : M \to \mathbf{R}^{m+1}$$

could be a proper mapping in terms of solutions of the d'Alembertian equations on M and $v_i \ge a_j^i \ge j$ on x_j^i.

As we did in Lemma $*$, the functions $v_1, \cdots, v_{m+1}, u_1, \cdots, u_{2m+1}$ together form a proper embedding of M into $\mathbf{R}^{3m+2}$. And, using the same projection techniques in Lemma $*$ successively, we can choose real numbers $b_{k,l}$ as close to zero as desired, in the strong topology of distributions, so that the mapping $w : M \to \mathbf{R}^{2m+1}$

$$\begin{aligned} p \to (v_1(p) - \sum_{l=1}^{m+1} b_{1,l} u_{m+l}(p), \ v_2(p) - \sum_{l=1}^{m+1} b_{2,l} u_{m+l}(p), \cdots, \\ v_{m+1}(p) - \sum_{l=1}^{m+1} b_{m+1,l} u_{m+l}(p), \ u_1(p) - \sum_{l=1}^{m+1} b_{m+2,l} u_{m+l}(p), \cdots, \\ u_m(p) - \sum_{l=1}^{m+1} b_{2m+1,l} u_{m+l}(p)) \end{aligned}$$

is an embedding of M into $\mathbf{R}^{2m+1}$ as we described in Lemma $*$. Moreover, this w is a proper mapping.

In fact, let $b_{k,l}$ as $|b_{k,l}| < \frac{1}{m+1}$ for all k, l. Then, for each i,

$$\begin{aligned} & v_i(p) - \sum_{l=1}^{m+1} b_{i,l} u_{m+l}(p) \\ & \ge a_j^i - (m+1) \frac{1}{m+1} \sup_{x_j^i} \max_{1 \le n \le 2m+1} |u_n(p)| \\ & \ge a_j^i - \frac{1}{2} a_j^i = \frac{1}{2} a_j^i \ge \frac{1}{2} j \end{aligned}$$

since $v_i(p) \geq a_j^i$ on x_j^i.
Letting j approach ∞, this mapping becomes a proper mapping of M by Lemma 2.

The proof of the theorem is complete.

Finally the next corollary can be proved easily from Theorem and the same method as the proof of Theorem **.

COROLLARY. *Any connected open Lorentzian manifold M of dimension m can be properly embedded into $\mathbf{R}_1^{p+1}$ $(p \geq 2m+1)$ by solutions of the d'Alembertian equation on M.*

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