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SOME RESULTS ON THE FOURIER-LAPLACE IMAGE OF SPECIAL CLASSES OF ω-TEMPERED DISTRIBUTION

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Schwartz [4] shows that the Fourier-Laplace image of a class of distributions which decrease exponentially with some exponent is a class of holomorphic functions which are analytic in a corresponding strip domain along the real axis. Here, we use the term "the Fourier-Laplace transform $F_u(\xi+i\eta)$ of a distribution $u(x)$ is defined by the Fourier transform of $e^{\eta x}u(x)$ when the distribution $e^{\eta x}u(x)$ belongs to S_ω'. Let $\Phi(x)$ be a function with a certain increasing condition, Gel'fand-Shilov [2] shows that the Fourier image of a class of C^∞-functions, any element of which can be estimated by $C\exp(-\Phi((1\pm\varepsilon)x))$ together with its derivatives, coincides with the class of entire function $F(\xi+i\eta)$, which satisfies the estimate $|(\xi+i\eta)^kF(\xi+i\eta)|\leq C_k\exp\Psi(\eta)$ for all k, where $\Psi(\eta)$ is the dual function of $\Phi(x)$ in the sense of Young. K. Hayakawa consider a class ${}_\Phi S'$ of distributions, which consists of all distributions of the type $e^{-\Phi(x)}u(x)$, where $u(x)$ is an element of S' and has a result: the Fourier-Laplace image of ${}_\Phi S'$ is nearly equal to the class of entire functions $F(\zeta)=F(\xi+i\eta)$ which satisfies the estimate

$$|F(\xi+i\eta)|\leq C(1+|\xi+i\eta|)^N e^{\psi(\eta)}$$

for an integer N.

In this paper, we treat this problem in S_ω'-Category such that $e^{-\Phi(|x|)+\eta x}$ belongs to S_ω for all $\eta\in R$ and get some results analogous to those results in [3]. Let Ω be an open interval in R. We define spaces ${}_\Phi S_\omega'$ and ${}_{\Phi,\Omega}S_\omega'$ as follows:

$${}_\Phi S_\omega'=\{u(x)\in\mathcal{D}_\omega' \mid e^{\Phi(|x|)}u(x)\in S_\omega'\}$$
$${}_{\Phi,\Omega}S_\omega'=\{u(x)\in\mathcal{D}_\omega' \mid e^{\lambda x+\Phi(|x|)}\ u(x)\in S_\omega' \text{ for all } \lambda \text{ in } \Omega\}$$

THEOREM 1. *The Fourier-Laplace transform $F_u(\xi+i\eta)$ of $u\in{}_\Phi S_\omega'$ satisfies*

$$|F_u(\xi+i\eta)|\leq Ce^{\Psi(|\eta+\varepsilon|)} \text{ for some } C>0$$

Proof. Since $e^{-\Phi(|x|)+\eta x}$ belongs to S_ω, $e^{\eta x}u(x)$ belongs to S_ω' and hence we can consider the Fourier-Laplace transform for an element of ${}_\Phi S_\omega'$.

$$|F_u(\xi+i\eta)|=|{}_{S_\omega'}\langle e^{\Phi(|x|)}u(x),e^{-\Phi(|x|)-i(\xi+i\eta)x}\rangle_{S_\omega}|$$
$$\leq CP_{0,\lambda}(e^{-\Phi(|x|)-i(\xi+i\eta)x})$$

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$$=C \sup e^{\lambda\omega(x)} |e^{-\Phi(|x|)+\eta x}|$$
$$\leq C \sup e^{-\phi(|x|)+(\eta+\varepsilon)x}$$
$$=Ce^{\Psi(|\eta+\varepsilon|)}$$

where we have taken ε such that $\lambda\omega(x) \leq \varepsilon|x|$.

EXAMPLE. If we take $\Phi(|x|)=\frac{1}{2}x^2$, then $\Psi(\eta)=\frac{1}{2}\eta^2$ and also $e^{-\Phi(|x|)+\eta x}$ belongs to S_ω. We have $|F_u(\xi+i\eta)| \leq C\, e^{\Psi(|\eta+\varepsilon|)}$ since $\sup e^{\lambda\omega(x)+\varepsilon x}|e^{-x^2/2+(\eta+\varepsilon)x}| \leq C\, e^{(\eta+\varepsilon)^2/2}$

THEOREM 2. *The Fourier-Laplace transferm $F_u(\xi+i\eta)$ of $u \in {}_{\Phi,\Omega}S_\omega'$ satisfies*

$$|F_u(\xi+i\eta)| \leq Ce^{\Psi(|\eta-\lambda-\varepsilon|)}$$

for some $C>0$ and for any $\lambda \in \Omega$.

Proof. Note that ${}_{\Phi,\Omega}S_\omega' \subset {}_{\Phi}S_\omega'$ since $S_\omega \cdot S_\omega' \subset S_\omega'$. So we can consider the Fourier-Laplace transform for an element of ${}_{\Phi,\Omega}S_\omega'$.

$$|F_u(\xi+i\eta)| = |{}_{S_\omega'}\langle e^{\lambda x+\Phi(|x|)}\ u(x),\ e^{-\Phi(|x|)-i(\xi+i\eta)x-\lambda x}\rangle_{S_\omega}|$$
$$\leq CP_{0,l}(e^{-\Phi(|x|)-i(\xi+i\eta)x-\lambda x})$$
$$\leq C \sup e^{l\omega(x)} |e^{-\phi(x)+\eta x-\lambda x}|$$
$$=Ce^{\Psi(|\eta-\lambda+\varepsilon|)}$$

where we have taken ε such that $l\omega(x) \leq \varepsilon|x|$.

REMARK. We know that $e^{-\Phi(|x|)+\eta x}$ belongs to S $(\supset S_\omega)$. We hope that $e^{-\Phi(|x|)+\eta x}$ belongs to S_ω automatically. But if we take $\Phi(|x|)=\frac{|x|^2}{2}$, $e^{-\Phi(|x|)+\eta x}$ belongs to S_ω.

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