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ON THE DOMAIN OF NULL-CONTROLLABILITY OF A LINEAR PERIODIC SYSTEM

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0. Introduction

In [1], E. B. Lee and L. Markus described a sufficient condition for which the domain of null-controllability of a linear autonomous system is all of R^n. The purpose of this note is to extend the result to a certain linear nonautonomous system. Thus we consider a linear control system

$$\frac{dx}{dt}=A(t)x+B(t)u$$

in the Eculidean n-space R^n where $A(\mathrm{t})$ and $B(t)$ are $n\times n$ and $n\times m$ matrices, respectively, which are continuous on $0\leq t<\infty$ and $A(t)$ is a periodic matrix of period ω. Admissible controls are bounded measurable functions defined on some finite subintervals of $[0, \infty)$ having values in a certain convex set Ω in R^m with the origin in its interior. And we present a sufficient condition for which the domain of null-controllability is all of R^n.

1. Preliminaries

Consider a linear control system in R^n

$$\frac{dx}{dt}=A(t)x+B(t)u \tag{1.1}$$

Where $A(t)$ and $B(t)$ are $n\times n$ and $n\times m$ matrices, respectively, which are continuous on $[0, \infty)$ but not necessarily periodic.

For any bounded measurable control $u(t)$ on $[t_0, t_1]$ $(t_0\geq 0)$ and for any initial state x_0 at t_0, (1.1) has a unique solution $x(t)$ with $x(t_0)=x_0$ existing on $[t_0, t_1]$ and this solution is given by

$$x(t)=F(t_0;t)x_0+F(t_0;t)\int_{t_0}^{t} F^{-1}(t_0;s)B(s)u(s)ds \tag{1.2}$$

where $F(t_0;t)$ is a fundamental matrix of the homogeneous system

$$\frac{dx}{dt}=A(t)x$$

with $F(t_0;t_0)=I$ (the identity matrix).

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DEFINITION 1. We say that the system (1.1) is completely controllable at $t_0 \geq 0$ if, for any x_0, x_1 in R^n there exists a bounded measurable control $u(t)$ on some finite interval $[t_0, t_1]$ with values in R^m such that.

$$x_1 = F(t_0;t_1)x_0 + F(t_0;t_1)\int_{t_0}^{t_1} F^{-1}(t_0;s)B(s)u(s)\ ds,$$

that is, there exists *a bounded measurable control on some finite interval* $[t_0, t_1]$ *which steers* x_0 *to* x_1.

It is well known that the system (1.1) is completely controllable at t_0 iff, for any $t_1 > t_0$, the matrix

$$M(t_0, t_1) = \int_{t_0}^{t_1} F^{-1}(t_0;t)B(t)B'(t)F^{-1'}(t_0;t)dt$$

is nonsingular where the prime denotes the transposed matrix.

2. Domain of Null-Controllability

Let Ω be a given convex set in R^m containing the origin in its interior and consider a linear control system in R^n

$$\frac{dx}{dt} = A(t)x + B(t)u, \quad u \in \Omega \tag{2.1}$$

where $A(t)$ and $B(t)$ are same as in (1.1) and admissible controls are bounded measurable functions on some finite subinterval of $[0, \infty)$ having values in Ω.

DEFINITION 2. The domain of null-controllability of (2.1) at $t_0 \geq 0$ is the set $C(t_0)$ of those points x_0 in R^n which can be steered to the origin by some admissible control; that is, there exists a bounded measurable control $u(t)$ on some interval $[t_0, t_1]$ having values in Ω such that

$$F(t_0;t_1)x_0 + F(t_0;t_1)\int_{t_0}^{t_1} F^{-1}(t_0;s)B(s)u(s)ds = 0$$

equivallently,

$$x_0 = -\int_{t_0}^{t_1} F^{-1}(t_0;s)B(s)u(s)\ ds$$

It is clear that $C(t_0)$ is convex and $0 \in C(t_0)$.

THEOREM 1. *The domain* $C(t_0)$ *of null-controllability at* $t_0 \geq 0$ *contains a neighborhood of the origin iff the system* (2.1) *is completely controllable at* t_0.

Proof, For $t_1 > t_0$, let

$$M(t_0, t_1) = \int_{t_0}^{t_1} F^{-1}(t_0;t)B(t)B'(t)F^{-1'}(t_0;t)\ dt$$

Then $M(t_0, t_1)$ is a symmetric matrix which is positive semidefinite.

Suppose that the system (2.1) is completely controllable at t_0. Then $M(t_0, t_1)$ is nonsingular for any $t_1 > t_0$. Choose any $t_1 > t_0$. Since $F^{-1}(t_0;t)$ and B(t) are continuous on $[t_0, t_1]$, there exists a constant $K_1>0$ such that $|F^{-1}(t_0;t)|\leq K_1$ and $|B(t)|\leq K_1$ for all $t_0\leq t\leq t_1$. *Let*

$$|M(t_0,t_1)|=K_2>0.$$

Choose $r>0$ so that $|u|<r$ implies $u\in\Omega$. Let x be any point in R^n such that $|x|<\frac{r}{K_1^2K_2}$. If we let, for $t_0\leq t\leq t_1$,

$$\xi=-M(t_0,t_1)^{-1}x,\ \ u(t)=B'(t)F^{-1'}(t_0;t)\xi$$

then

$$|u(t)|=|B'(t)F^{-1'}(t_0;t)\xi|\leq K_1^2K_2|x|<r$$

so that $u(t)\in\Omega$ for all $t_0\leq t\leq t_1$. Moreover,

$$-\int_{t_0}^{t_1}F^{-1}(t_0;t)B(t)u(t)\,dt=-\int_{t_0}^{t_1}F^{-1}(t_0;t)B(t)B'(t)F^{-1'}(t_0;t)\xi\,dt$$
$$=-M(t_0,t_1)\xi=x.$$

Thus $x\in C(t_0)$; that is $C(t_0)$ contains the set

$$\left\{x\in R^n;\,|x|<\frac{r}{K_1^2K_2}\right\}$$

Conversely, suppose $C(t_0)$ does not contain a neighborhood of the origin. Then $C(t_0)$ lies on a hyperplane passing through the origin. Thus there exists a nonzero vector ζ in R^n such that, for any $t_1>t_0$ and for any admissible control $u(t)$ on $[t_0,t_1]$,

$$\int_{t_0}^{t_1}\zeta F^{-1}(t_0;t)B(t)u(t)\,dt=0.$$

For any $t_1>t_0$, consider the control $u(t)=B'(t)F^{-1'}(t_0;t)\zeta'$ on $[t_0, t_1]$. For $|\zeta|$ sufficiently small $u(t)\in\Omega$ for all $t_0\leq t\leq t_1$ and we must have

$$\int_{t_0}^{t_1}\zeta F^{-1}(t_0;t)B(t)B'(t)F^{-1'}(t_0;t)\zeta'\,dt$$
$$=\zeta M(t_0,t_1)\zeta'=0.$$

Thus $M(t_0,t_1)$ is singular so that system (2.1) is not completely controllable at t_0.

3. Linear Periodic System

Now consider a linear periodic system in R^n

$$\frac{dx}{dt}=A(t)x+B(t)u,\ \ u\in\Omega \tag{3.1}$$

where $A(t)$, $B(t)$ and Ω are same as in (2.1) and, in addition, we assume that $A(t)$ is a periodic matrix of period ω on $[0,\infty)$.

If $F(t)$ is a fundamental matrix of the corresponding homogeneous system

$$\frac{dx}{dt}=A(t)x \tag{3.2}$$

then, by Floquet's theorem, there exists a periodic nonsingular matrix $P(t)$ of period ω and a constant matrix R such that $F(t)=P(t)\exp(tR)$. We call the eigenvalues of the nonsingular matrix $\exp(\omega R)$ the multipliers of the system (3.2) and the eigenvalues of the matrix R are called the characteristic exponents of the system (3.2).

Following lemma is well know.

LEMMA. *If all the characteristic exponents of the system* (3.2) *have negative real parts, then the zero solution of* (3.2) *is asymptotically stable.*

Combining Theorem 1 and the above lemma, we obtain the following theorem which is the main result.

THEOREM 2. *For the linear periodic system*(3.1), *suppose*

(1) 0 *is in the interior of* Ω

(2) *system*(3.1) *is completely controllable at every* $t' \geq t_0$

(3) *characteristic exponents of* (3.2) *have negative real parts.*

Then the domain $C(t_0)$ *of null-controllability of* (3.1) *at* t_0 *is all of* R^n.

Proof. Let x_0 be any point in R^n and choose the control $u(t)$ which identically zero for all $t\geq 0$. If $x(t)$ is the solution of (3.1) corresponding to $u(t)$ with $x(t_0)=x_0$, then, by condition(3), $x(t')$ belongs to $C(t')$ for sufficiently large t'. But then there exists an admissible control $v(t)$ on some interval $[t', t_1]$ which steers $x(t')$ to the origin.

References

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