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On the Law of the Iterated Logarithm without Assumptions about the Existence of Moments for the Sums of Sign-Invariant Random Variables

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ABSTRACT

Petrov (1968) gave two theorems on the law of the iterated logarithm without any assumptions about the existence of moments of independent random variables. In the present paper we show that the same holds true for sign-invariant random variables.

1. Introduction

Most of the known results concerning the law of the iterated logarithm and its generalizations are obtained for a sequence of independent random variables which have finite variances or which are even bounded. Petrov (1968) proved two theorems on the law of the iterated logarithm without any assumptions about the existence of moments of independent random variables. In the present paper I shall prove same theorems for the case of sign-invariant random variables.

Definition. The sequence $\{X_n\}$ is sign-invariant if every finite dimensional d.f. of the sequence is invariant under any changes in the signs of the $\{X_n\}$.

It is obvious that a sequence of independent random variables with a common d.f. F(x) is sign-invariant if and only if F(x) is symmetric, i.e., every one-dimensional d.f. is invariant under changes in signs.

Now we consider the following model which is useful to study sign-invariant random variables: Let $\mathcal{F}$ be a set of distributions on $\mathcal{R}^1$ with the topology of weak

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convergence and let $\mathcal{A}$ be the σ-field generated by the open sets. We denote by $\mathcal{F}_1^\infty$ the space consisting of all infinite sequence $(F_1, F_2, \cdots)$, $F_n \in \mathcal{F}$ and $\mathcal{R}_1^\infty$ the space consisting of all infinite sequences $(x_1, x_2, \cdots)$ of real numbers. Take the σ-field $\mathcal{A}_1^\infty$ to be the smallest σ-field of subsets of $\mathcal{F}_1^\infty$ consising all finite - dimensional rectangles and take $\mathcal{B}_1^\infty$ to be the $\mathcal{B}$orel σ-field of $\mathcal{R}_1^\infty$. Let $w = (F_1^w, F_2^w, \cdots)$ be the coordinate process in $\mathcal{F}_1^\infty$ and ν its distribution on $\mathcal{A}_1^\infty$. For each w define a probability measure P_w on $(\mathcal{R}_1^\infty, \mathcal{B}_1^\infty)$ so that $P_w = \prod_{i=1}^\infty F_i^\infty$. A monotone class argument shows that $P_w(B)$, $B \in \mathcal{B}_1^\infty$, is $\mathcal{A}_1^\infty$- measurable as a function of w. so we can define a new probability measure such that $\tilde{P}(B) = \int P_w(B)\nu(dw)$. Define the process X_n on $(\mathcal{R}_1^\infty, \mathcal{B}_1^\infty)$ such that $X_n(x_1, x_2, \cdots) = x_n$ and set $S_n = X_1 + X_2 + \cdots + X_n$. By the definition of P_w, X_n are independent with respect to P_w.

Definition. Let $(\mathcal{R}_1^\infty, \mathcal{B}_1^\infty, \hat{P})$ be a probability space. Then $\hat{P}$ is called sign-invariant if $\{X_n\}$ is sign-invariant.

The following is due to Hong (1990).

Theorem 0. Let $\hat{P}$ be any sign-invariant probability measure on $(\mathcal{R}_1^\infty, \mathcal{B}_1^\infty)$. Then one can determine ν on $\mathcal{F}_1^\infty$ where $\mathcal{F} = \{\frac{1}{2}(\delta_y + \delta_{-y}) \mid y \in \mathcal{R}^+ \cup \{0\}\}$ so that $\hat{P} = \tilde{P}$ and P_k, the k-th marginal of $\hat{P}$, is given by $\hat{P}_k = \int F_k^w \nu(dw)$, $k = 1, 2, \cdots$.

2. Results

Theorem 1. Suppose P be a sign-invariant probability measure on $(\mathcal{R}_1^\infty, \mathcal{B}_1^\infty)$ and $\{X_n\}$ is pairwise independent. If there exists a sequence of number $\{B_n\}$ such that

$$B_n \uparrow \infty,$$
$$\frac{B_{n+1}}{B_n} \longrightarrow 1$$

and

$$\sup_x \mid P\{S_n < B_n^{\frac{1}{2}} x\} - (2\pi)^{-\frac{1}{2}} \int_{-\infty}^x e^{-\frac{t^2}{2}} dt \mid = O[(log B_n)^{-1-\delta}]$$

for some $\delta > 0$ as $n \longrightarrow \infty$, then

$$P\{\overline{\lim} \frac{S_n}{(2B_n log log B_n)^{\frac{1}{2}}} = 1\} = 1.$$

Theorem 1 is a generalization of a result of Petrov (1968).

Theorem 2. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of positive numbers such that $a_n < c_n$, $c_n \uparrow \infty$ and there exists $\lim(b_n/c_n) \doteq \gamma$, say. We set

$$Y_n = \begin{cases} X_n & \text{if } |X_n| < a_n \\ 0 & \text{if } |X_n| \geq a_n. \end{cases}$$

Let

$$P\{\overline{\lim} \frac{1}{b_n} \sum_{k=1}^{n} Y_k = 1\} = 1.$$

If

$$\sum_{n=1}^{\infty} \int_{|x| \geq a_n} \frac{x^2}{x^2 + c_n^2} \, dP\{X_n \leq x\},$$

then

$$P\{\overline{\lim} \frac{1}{c_n} \sum_{k=1}^{n} X_k = \gamma\} = 1.$$

Proof of Theorem 1. Just follow the proof of [5, Theorem 1] using the following lemma and the Borel-Cantelli lemma for pairwise independent event ([2, P76]).

Lemma. Let ν be a probability measure on $\mathcal{F}_1^\infty$ where $\mathcal{F} = \{\frac{1}{2}(\delta_y + \delta_{-y} \mid y \in \mathcal{R}^+ \cup \{0\}\}$. Then we have

$$\tilde{P}\{\max_{1 \leq k \leq n} S_k \geq x\} \leq 2\tilde{P}\{S_n \geq x\}.$$

Proof. Since $\{X_n\}$ is independent with respect to P_w for all w and each has symmetric distribution, by Levy inequality for all w and for every x,

$$P_w\{\max_{1 \leq k \leq n} S_k \geq x\} \leq 2P_w\{S_n \geq x\}.$$

Then we have

$$\begin{aligned} \tilde{P}\{\max_{1 \leq k \leq n} S_k \geq x\} &= \int P_w\{\max_{1 \leq k \leq n} S_k \geq x\} \, \nu(dw) \\ &\leq 2 \int P_w\{\max_{1 \leq k \leq n} S_k \geq x\} \, \nu(dw) \\ &= 2\tilde{P}\{S_n \geq x\}. \end{aligned}$$

Proof of Theorem 2. By Theorem 0, we have a probability measure ν on $\mathcal{F}_1^\infty$ such that $\tilde{P} = P$ and $P\{X_n \le t\} = \int F_k^w(t)\ \nu(dw)$, $k = 1, 2, \ldots$. Now

$$\begin{aligned}
\int \sum_{n=1}^{\infty} \int_{|x| \ge a_n} & \frac{x^2}{x^2 + c_n^2}\ dP_w\{X_n \le t\}\ \nu(dw) \\
&= \int \sum_{n=1}^{\infty} \int_{|x| \ge a_n} \frac{x^2}{x^2 + c_n^2}\ dF_n^w(x)\ \nu(dw) \\
&= \sum_{n=1}^{\infty} \int \int_{|x| \ge a_n} \frac{x^2}{x^2 + c_n^2}\ dF_n^w(x)\ \nu(dw) \\
&= \sum_{n=1}^{\infty} \int_{|x| \ge a_n} \frac{x^2}{x^2 + c_n^2}\ dP\{X_n \le t\} \ < \ \infty\ ,
\end{aligned}$$

by assumption. Therefore, we have for $\nu - a.e.$ w

$$\sum_{n=1}^{\infty} \int_{|x| \ge a_n} \frac{x^2}{x^2 + c_n^2} dP_w\{X_n \le t\} \ < \ \infty.$$

Since $\{X_n\}$ is independent with respect to P_w for all w, according to Theorem 2 [5], we have for $\nu - a.e.$ w

$$P_w\{\overline{\lim}\ \frac{1}{c_n} \sum_{k=1}^{n} X_k \ = \ \gamma\} \ = \ 1.$$

Hence we have

$$P\{\overline{\lim}\ \frac{1}{c_n} \sum_{k=1}^{n} X_k \ = \ \gamma\} \ = \ \int P_w\{\overline{\lim}\ \frac{1}{c_n} \sum_{k=1}^{n} X_k \ = \ \gamma\}\ \nu(dw) \ = \ 1.$$

This completes the proof.

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