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GENERALIZATIONS OF ISERMANN'S RESULTS IN VECTOR OPTIMIZATION

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1. Introduction

Vector optimization problems consist of two or more objective functions and constraints. Optimization entails obtaining efficient solutions. Geoffrion [3] introduced the definition of the properly efficient solution in order to eliminate efficient solutions causing unbounded trade-offs between objective functions.

In 1974, Isermann [7] obtained a necessary and sufficient condition for an efficient solution of a linear vector optimization problem with linear constraints and showed that every efficient solution is a properly efficient solution. Since then, many authors [1, 2, 4, 5, 6] have extended the Isermann's results. In particular, Gulati and Islam [4] derived a necessary and sufficient condition for an efficient solution of a linear vector optimization problem with nonlinear constraints, under certain assumptions.

In this paper, we consider the following nonlinear vector optimization problem (NVOP):

$$\text{(NVOP)} \qquad \begin{array}{l} \text{Maximize } f(x) = [f_1(x), \cdots, f_p(x)]^t \\ \text{subject to } x \in X = \{x : g(x) \geqq 0\}, \end{array}$$

where for each i, f_i is a differentiable function from R^n into R and g is a differentiable function from R^n into R^m.

Recently, Gulati and Islam [6] obtained a necessary and sufficient condition for an efficient solution of (NVOP), under certain assumptions.

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We derive a necessary and sufficient condition for an efficient solution of (NVOP), under certain assumptions different from Gulati and Islam's [6]. Our results generalize those of Isermann [7], and Gulati and Islam [4], who deal with linear vector optimization problems.

2. Results

We first recall the following definitions.

DEFINITION 2.1. $x^0 \in X$ is said to be an efficient solution of (NVOP) if there is no $x \in X$ such that $f(x) \geqq f(x^0)$ and $f(x) \neq f(x^0)$.

DEFIITION 2.2 [3]. $x^0 \in X$ is said to be a properly efficient solution of (NVOP) if it is efficient and there exists a scalar $M > 0$ such that, for each i, we have

$$[f_i(x) - f_i(x^0)] \ / \ [f_j(x^0) - f_j(x)] \leqq M$$

for some j such that $f_j(x) < f_j(x^0)$ whenever $x \in X$ and $f_i(x) > f_i(x^0)$.

LEMMA 2.1. *$x^0 \in X$ is an efficient solution of (NVOP) if and only if x^0 is an optimal solution of the following scalar optimization problem (SOP):*

(SOP)

$$\begin{aligned} &\textit{Maximize} \quad d^t f(x) \\ &\textit{subject to} \quad x \in X_s = \{x : g(x) \geqq 0, f(x) - f(x^0) \geqq 0\}, \end{aligned}$$

where d is a positive constant in R^p.

Proof. Let $x^0 \in X$ be an efficient solution of (NVOP). Suppose that x^0 is not an optimal solution of (SOP). Then there exists x^* such that

$$g(x^*) \geqq 0, \ \ f(x^*) - f(x^0) \geqq 0 \ \text{ and } \ d^t f(x^*) > d^t f(x^0).$$

Since $d > 0$, we have

$$g(x^*) \geqq 0, \ \ f(x^*) \geqq f(x^0) \ \text{ and } \ f(x^*) \neq f(x^0).$$

This contradicts the fact that x^0 is an efficient solution of (NVOP).

Conversly, let x^0 be an optimal solution of (SOP). Suppose that x^0 is not an efficient solution of (NVOP). Then there exists $\overline{x} \in X$ such that

$$g(\overline{x}) \geqq 0, \quad f(\overline{x}) \geqq f(x^0) \text{ and } \quad f(\overline{x}) \neq f(x^0).$$

Since $d > 0$, we have

$$g(\overline{x}) \geqq 0, \quad \text{and} \quad d^t f(\overline{x}) > d^t f(x^0).$$

This contradicts the fact that x^0 is an optimal solution of (SOP).

Whenever we assume that the set X_s satisfies a constraint qualification, we shall mean that the set X_s satisfies the Kuhn - Tucker constraint qualification or the Arrow - Hurwicz - Uzawa constraint qualification defined in [9].

THEOREM 2.1 (ISERMANN TYPE NECESSARY CONDITION). *Suppose that g_I is quasiconcave at $x^0 \in X$, where $I = \{i : g_i(x^0) = 0\}$ and the set X_s satisfies a constraint qualification at x^0. If x^0 is an efficient solution of (NVOP), then there exists $u^0 \in R^p$ such that $u^0 > 0$ and*

$$\textit{for all } x \in X, \quad {u^0}^t \nabla f(x^0) x \leqq {u^0}^t \nabla f(x^0) x^0.$$

Proof. Suppose that x^0 is an efficient solution of (NVOP). By Lemma 2.1, x^0 is an optimal solution of (SOP) for a fixed $d > 0$. Since the set X_s satisfies a constraint qualification at x^0, by the Kuhn - Tucker necessary optimality theorem [8, 9], there exist $v^0 \geqq 0, \ v^0 \in R^p, \ w^0 \geqq 0, \ w^0 \in R^n$ such that

$$d^t \nabla f(x^0) + {v^o}^t \nabla f(x^0) + {w_I^0}^t \nabla g_I(x^0) = 0.$$

Let $u^0 = d + v^0$. Then $u^0 > 0$ and

$$(1) \qquad {u^o}^t \nabla f(x^0) + {w_I^0}^t \nabla g_I(x^0) = 0.$$

Since for all $x \in X$, $g_I(x) \geqq g_I(x^0)$, by the quasiconcavity of g_I at x^0,

$$\nabla g_I(x^0)(x - x^0) \geqq 0,$$

and

$$w_I^{0^t} \nabla g_I(x^0)(x - x^0) \geqq 0 \text{ for all } x \in X.$$

From (1), we have, for all $x \in X$

$$u^{0^t} \nabla f(x^0)x \leqq u^{0^t} \nabla f(x^0)x^0.$$

REMARK 2.1. Gulati and Islam [6] showed that when f is pseudoconvex at $x^0 \in X$, g_I is quasiconcave at x^0 and the set $\{x : g(x) \geqq 0, \nabla f(x^0)x - \nabla f(x^0)x^0 \geqq 0\}$ satisfies a constraint qualification at x^0, the result of Theorem 2.1 holds. In Theorem 2.1, the condition of pseudoconvexity of f is removed.

THEOREM 2.2 [6](ISERMANN TYPE SUFFICIENT CONDITION). *Suppose that f is pseudoconcave at $x^0 \in X$. If there exists $u^0 \in R^p$ such that $u^0 > 0$ and for all $x \in X$,*

$$u^{0^t} \nabla f(x^0)x \leqq u^{0^t} \nabla f(x^0)x^0,$$

then x^0 is an efficient solution of (NVOP).

Hence we have the following theorem from above theorems.

THEOREM 2.3. *Assume that f is pseudoconcave at $x^0 \in X$, g_I is quasiconcave at x^0, where $I = \{i : g_i(x^0) = 0\}$ and the set X_s satisfies a constraint qualification at x^0. Then x^0 is an efficient solution of (NVOP) if and only if there exists $u^0 \in R^p$ such that $u^0 > 0$ and for all $x \in X$,*

$$u^{0^t} \nabla f(x^0)x \leqq u^{0^t} \nabla f(x^0)x^0.$$

We give an example satisfying all the assumptions of Theorem 2.3.

EXAMPLE 2.1 [8,10]. Consider the following vector optimization problem:

$$\begin{aligned} &\text{Maximize } f(x) = (x, -x^2 + 2x)^t \\ &\text{subject to } x \in Y = \{x \in R : g(x) = -x^2 + 2x \geqq 0\}. \end{aligned}$$

We can see easily that the set of all the efficient solutions consists of the line segment [1,2], f is pseudoconcave and g is quasiconcave. Let $x^0 \in (1,2]$. Then the set $\{x \in R : g(x) \geqq 0,\ f(x) - f(x^0) \geqq 0\}$ satisfies the Kuhn - Tucker constraint qualification at x^0. Hence all the assumptions of Theorem 2.3 are satisfied. In fact, $x^0 = 1$ is not a properly efficient solution.

From Theorem 2.3, and Theorem 1 and 2 in [3], we have the following corollary.

COROLLARY 2.1. *Suppose that f is pseudoconcave at $x^0 \in X$, g_I is quasiconcave at x^0, where $I = \{i : g_i(x^0) = 0\}$ and the set X_s satisfies a constraint qualification at x^0. Then x^0 is an efficient solution of (NVOP) if and only if x^0 is a properly efficient solution of the following linear vector optimization problem:*

$$\text{(LIVOP)} \quad \begin{aligned} &\text{Maximize } [\nabla f_1(x^0)x, \cdots, \nabla f_p(x^0)x]^t \\ &\text{subject to } g(x) \geqq 0 \end{aligned}$$

REMARK 2.2. Our Theorem 2.3 and Corollary 2.1 are generalizations of Theorem 2 and 4 in [4], respectively.

Isermann [7] considered the following linear vector optimization problem (LVOP):

$$\text{(LVOP)} \qquad \begin{aligned} &\text{Maximize } Cx \\ &\text{subject to } Ax = b, x \geqq 0, \end{aligned}$$

where C is an $p \times n$ matrix, A is an $m \times n$ matrix, $x \in R^n$ and $b \in R^m$.

We can rewtite (LVOP) as follows:

(LVOP')

$$\begin{aligned} &\text{Maximize } Cx \\ &\text{subject to } Ax \geqq b \\ &\qquad -Ax \geqq -b \\ &\qquad x \geqq 0. \end{aligned}$$

Since $(\mathrm{LVOP})'$ satisfies all the assumptions of Theorem 2.3 at all the efficient solutions of (LVOP), by Theorem 2.3 and Corollary 2.1, we obtain the following corollary.

COROLLARY 2.2 [7]. *The following statements are equivalent.*

(1) A feasible point x^0 is an efficient solution of (LVOP).

(2) x^0 is a feasible point and there exists $u^0 \in R^p$ such that $u^0 > 0$ and for all feasible points x of (LVOP),

$$u^{0^t} Cx \leqq u^{0^t} Cx^0.$$

(3) A feasible point x^0 is a properly efficient solution of (LVOP).

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