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NORMAL HOLONOMY GROUP OF A RIEMANNIAN FOLIATION*

HONG KYUNG PAK AND JIN SUK PAK

1. Introduction

Recently Olmos ([4]) studied some properties about the restricted normal holonomy group of a submanifold isometrically immersed in a space of constant curvature. Let $\mathcal{V}$ be a foliation on a Riemannian manifold (M, g). Each leaf $\mathcal{L}$ of $\mathcal{V}$ with the induced metric can be considered as an isometric immersed submanifold of (M, g). For the normal connection D defined on the normal bundle Q of $\mathcal{V}$ we define the normal holonomy group $\mathcal{H}_x^D(\mathcal{L})$ of $\mathcal{V}$ along a leaf $\mathcal{L}$ through x as the group of parallel displacements P_c along loops $c : [0,1] \longrightarrow \mathcal{L}, c(0) = c(1) = x$.

Associated to a foliation $\mathcal{V}$ is the linear holonomy group $Hol_x(\mathcal{L})$ of $\mathcal{V}$ along a leaf $\mathcal{L}$ through x, which is defined as a homomorphic image of the fundamental group of $\mathcal{L}$, hence a discrete group. Therefore it may be interesting to investigate the condition when the normal holonomy group of $\mathcal{V}$ along $\mathcal{L}$ is discrete.

In this paper, we will discuss on the above problem for the case that $\mathcal{V}$ is a Riemannian foliation. If $\mathcal{V}$ is a Riemannian foliation on (M, g), we derive some basic relations between the curvature R^D of the normal connection D and the curvature R of the Levi-Civita connection ∇ on (M, g) (see Lemma 1). From those formulas we will establish a sufficient condition, that is,

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MAIN THEOREM. *Let $\mathcal{V}$ be a Riemannian foliation on a connected Riemannian manifold (M,g) of constant curvature. If $\mathcal{V}$ is totally umbilic then the normal holonomy group $\mathcal{H}^D(\mathcal{L})$ of $\mathcal{V}$ along any leaf $\mathcal{L}$ is discrete.*

2. Proof of main theorem and other results.

Let $\mathcal{V}$ be a p-dimensional smooth foliation of codimension q on a connected Riemannian manifold (M, g). It is given by an exact sequence of vector bundles

$$0 \longrightarrow \mathcal{V} \longrightarrow TM \xrightarrow{\pi} Q \longrightarrow 0,$$

where Q is the normal bundle of $\mathcal{V}$. The metric g determines an orthogonal splitting $TM = \mathcal{V} \oplus \mathcal{H}$, where $\mathcal{H} = \mathcal{V}^{\perp}$. Hereafter, we denote by $\Gamma(\ )$ the space of all sections of a vector bundle ( ).

Let ∇ be the Levi-Civita connection on (M, g) and R its curvature tensor. There are two natural integrability tensors T and A globally defined by

$$T_E F := \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F,$$
$$A_E F := \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F$$

for arbitrary vector fileds E and F on M. Define a connection D: $\Gamma(TM)$ $\times\Gamma(Q) \longrightarrow \Gamma(Q)$ by

(1)

$$D_E\eta := \pi(\nabla_E\mathcal{H}\tilde{\xi})\ \ E \in \Gamma(TM),\ \ \eta \in \Gamma(Q), \tilde{\xi} \in \Gamma(TM) \text{ with } \pi(\tilde{\xi}) = \eta.$$

It is easy to verify that D is well-defined, which is called the normal connection of the foliation $\mathcal{V}$. Identifying $\mathcal{H}$ with Q by an isometric splitting $\sigma : (Q, g_Q := \sigma^* g_{\mathcal{H}}) \longrightarrow (\mathcal{H}, g_{\mathcal{H}})$, we can express simply (1) as

(2) $$D_E\eta = \mathcal{H}\nabla_E\eta\ \ E \in \Gamma(TM),\ \ \eta \in \Gamma(\mathcal{H}).$$

REMARK. The normal connection D is metrical with respect to $g_{\mathcal{H}}$ and has torsion $T^D(X,\eta) = A_\eta X$, for $X \in \Gamma(\mathcal{V})$, $\eta \in \Gamma(\mathcal{H})$, where the torsion of D is defined by $T^D(E,F) := D_E\mathcal{H}F - D_F\mathcal{H}F - D_F\mathcal{H}E - \mathcal{H}[E,F], E,F \in \Gamma(TM)$.

Recall ([2]) that $\mathcal{V}$ is Riemannian if and only if there exists an orthonormal adapted frame $\{X_i, \eta_a\}$, $1 \le i \le p$, $p+1 \le a \le p+q$, to $\mathcal{V}$ such that $\overset{\circ}{\nabla}_X \eta_a = 0$ for any $X \in \Gamma(\mathcal{V}\mid_U)$ in each flat chart U. Here $\overset{\circ}{\nabla}$ is the Bott connection ([1]) defined by $\overset{\circ}{\nabla}_E \eta := \mathcal{H}[\mathcal{V}E,\eta] + \mathcal{H}\nabla_{\mathcal{H}E}\eta$ for all $E \in \Gamma(TM)$, $\eta \in \Gamma(\mathcal{H})$. Note that the normal connection D coinsides with the Bott connection $\overset{\circ}{\nabla}$ if and only if $T^D = 0$, or equivalently $\mathcal{H}$ is integrable.

LEMMA 1. *If $\mathcal{V}$ is a Riemannian foliation on (M,g) then the curvature tensor R^D of D satisfies for any $X,Y \in \Gamma(\mathcal{V})$ and any $\xi,\eta,\psi \in \Gamma(\mathcal{H})$*

$$g(R^D(X,Y)\xi,\eta) = -g(R(X,Y)\xi,\eta) + g([\alpha_\xi,\alpha_\eta](X),Y), \tag{3}$$

(4)

$$g(R^D(X,\xi)\eta,\psi) = g(R(\eta,\psi)\xi,X) + g(A_\psi\xi,\alpha_\eta(X)) - g(A_\eta\xi,\alpha_\psi(X)),$$

where α is the shape operator associated to T which is characterized by the relation $g(\alpha_\xi(X),Y) = -g(\xi,T_XY)$.

Proof. Since those formulas are tensor equations, it suffices to prove them for $X,Y \in \Gamma(\mathcal{V})$ and for $\xi,\eta,\psi \in \mathcal{B}$. Here $\mathcal{B}$ denotes the space of all basic vector fields ξ, i.e., $\xi \in \Gamma(\mathcal{H})$ satisfying the condition $\overset{\circ}{\nabla}_X \xi = 0$ for any $X \in \Gamma(\mathcal{V})$. Since $[\Gamma(\mathcal{V}),\mathcal{B}] \subset \Gamma(\mathcal{V})$, we have

$$D_X\xi = A_\xi X forX \in \Gamma(\mathcal{V}), \xi \in \mathcal{B}.$$

On the other hand the following equations are well-known ([5]) :

$$\begin{aligned} g(R(X,Y)\xi,\eta) =& g((\nabla_X A)_\xi\eta,Y) - g((\nabla_Y A)_\xi\eta,X) + g(A_\xi X,A_\eta Y) \\ &- g(A_\xi Y,A_\eta X) - g(T_X\xi,T_Y\eta) + g(T_Y\xi,T_X\eta), \end{aligned}$$

$$\begin{aligned}g(R(\eta,\psi)\xi,X) =& g((\nabla_\xi A)_\eta\psi,X)+g(A_\eta\psi,T_X\xi)\\ &-g(A_\psi\xi,T_X\eta)-g(A_\xi\eta,T_X\psi).\end{aligned}$$

Now a direct computation yields

$$\begin{aligned}g(R^D(X,Y)\xi,\eta) &= g(D_XD_Y\xi-D_YD_X\xi-D_{[X,Y]}\xi,\eta)\\ &= g(D_XA_\xi Y-D_YA_\xi X-A_\xi[X,Y],\eta)\\ &= g(\nabla_XA)_\xi Y-(\nabla_YA)_\xi X+A_{A_\xi X}Y-A_{A_\xi Y}X,\eta)\\ &= -g((\nabla_XA)_\xi\eta,Y)+g((\nabla_YA)_\xi\eta,X)\\ &\qquad -g(A_\xi X,A_\eta Y)+g(A_\xi Y,A_\eta X)\\ &= -g(R(X,Y)\xi,\eta)-g(T_X\xi,T_Y\eta)+g(T_Y\xi,T_X\eta)\\ &= -g(R(X,Y)\xi,\eta)+g([\alpha_\xi,\alpha_\eta](X),Y).\end{aligned}$$

In the similar way

$$\begin{aligned}g(R^D(X,\xi)\eta,\psi) &= g(D_XD_\xi\eta-D_\xi D_X\eta-D_{[X,\xi]}\eta,\psi)\\ &= g(A_{\mathcal{H}\nabla_\xi\eta}X-\mathcal{H}\nabla_\xi A_\eta X-A_\eta[X,\xi],\psi)\\ &= -g((\nabla_\xi A)_\eta X+A_\eta\nabla_X\xi,\psi)\\ &= g((\nabla_\xi A)_\eta\psi,X)+g(A_\eta\psi,T_X\xi)\\ &= g(R(\eta,\psi)\xi,X)+g(A_\psi\xi,\alpha_\eta(X))-g(A_\eta\xi,\alpha_\psi(X)).\end{aligned}$$

Let $\mathcal{V}$ be a Riemannian foliation on (M,g). Let $X\in\Gamma(\mathcal{V})$ and $\xi\in\Gamma(\mathcal{H})$. For a tangent vector $X(x)\in\mathcal{V}_x$, $D_{X(x)}\xi$ given by

$$D_{X(x)}\xi:=(D_X\xi)(x)\ (x\in\mathcal{L})$$

is an element of $\mathcal{H}^{\mathcal{L}}_x$, where $\mathcal{H}^{\mathcal{L}}_x$ is the induced bundle on the leaf $\mathcal{L}$ through x by the inclusion $\iota:\mathcal{L}\longrightarrow M$.

If $c:[0,1]\longrightarrow\mathcal{L}$ is a curve in $\mathcal{L}$, a section $\xi\in\Gamma(\mathcal{H}^{\mathcal{L}})$ along c is said to be parallel along c if

$$D_{\dot{c}(t)}\xi=0\ (0\le t\le 1),$$

where $\dot{c}(t)$ denotes the tangent vector of c at $c(t)$. For a given element $\xi_0 \in \mathcal{H}^{\mathcal{L}}_{c(0)}$ it extends uniquely to a parallel section ξ along c with $\xi(0) = \xi_0$. Thus we have a linear isomorphism

$$\mathcal{H}^{\mathcal{L}}_{c(0)} \longrightarrow \mathcal{H}^{\mathcal{L}}_{c(1)} : \xi_0 = \xi(0) \longrightarrow \xi(1),$$

which is called the parallel displacemant of ξ_0 along c . In particular, the parallel displacement along a loop c at $x = c(0)$ induces an automorphism of $\mathcal{H}^{\mathcal{L}}_x$, which is written by P_c.

Denoting by $C(x)$ the loop space in a leaf $\mathcal{L}$ at x, the set $\mathcal{H}^D(\mathcal{L}) := \{P_c : c \in C(x)\}$ forms a group, called the normal holonomy group of $\mathcal{V}$ along $\mathcal{L}$ (at x). Since D is metrical with respect to $g_{\mathcal{H}}, \mathcal{H}^D(\mathcal{L})$ is a Lie subgroup of the orthogonal group $O(\mathcal{H}^{\mathcal{L}}_x)$

Let D be the normal connection (given in page 2) on the horizontal bundle $\mathcal{H}$ over (M,g) and ω be the connection on the associated principal $O(q)$-bundle $P(\mathcal{H}) \longrightarrow M$ with curvature form Ω.

Consider the induced bundle $p : P(\mathcal{H})^{\mathcal{L}} \longrightarrow L$ over a leaf $\mathcal{L}$ through $x \in M$. For the loop space $C(x)$ in $\mathcal{L}$ at x, the holonomy group of D at u_0 with $p(u_0) = x$ is defined by

$$\Phi^D(\mathcal{L}) := \{a \in O(q) : P_c(u) = ua\}.$$

This group can be identified with the normal holonomy group $\mathcal{H}^D(\mathcal{L})$ of $\mathcal{V}$ along $\mathcal{L}$. Taking account of the theorem of Ambrose-Singer, we have

LEMMA 2. *The Lie algebra $\underline{\mathcal{H}}^D(\mathcal{L})$ of the normal holonomy group $\mathcal{H}^D(\mathcal{L})$ along a leaf $\mathcal{L}$ is exactly the subalgebra of $so(q)$ spanned by all elements of the form $\Omega_u(\tilde{X},\tilde{Y})$, where $\tilde{X},\tilde{Y}$ are horizontal vectors at u and u is arbitrary point of $P(\mathcal{H})^{\mathcal{L}}$ which can be joined to u_0 by a horizontal curve. The curvature form Ω is locally expressed as*

$$(s^*\Omega)(X,Y) = R^D(s_*X, s_*Y) \quad X,Y \in \mathcal{V}_x$$

for a local section s of $P(\mathcal{H})^{\mathcal{L}}$.

From now on we prove the Main Theorem. First, recall that a foliation $\mathcal{V}$ on (M,g) is said to be totally umbilic if there exists a vector field $\nu \in \Gamma(\mathcal{H})$ such that

$$(5) \qquad T_X Y = g(X,Y)\nu \text{ for all } X, Y \in \Gamma(\mathcal{V}).$$

Then by (5), for $Y, Z \in \Gamma(\mathcal{V})$ and $\xi, \eta \in \Gamma(\mathcal{H})$ we have

$$\begin{aligned} g([\alpha_\xi, \alpha_\eta](Y), Z) &= g(T_Y\eta, T_Z\xi) - g(T_Y\xi, T_Z\eta) \\ &= \sum_{i=1}^{p}\{g(T_Y X_i, \eta)g(T_Z X_i, \xi) - g(T_Y X_i, \xi)g(T_Z X_i, \eta)\} \\ &= g(Y,Z)g(\nu,\xi)g(\nu,\eta) - g(Z,Y)g(\nu,\eta)g(\nu,\xi) = 0, \end{aligned}$$

hence by (3)

$$g(R^D(Y,Z)\xi,\eta) = -g(R(Y,Z)\xi,\eta).$$

Moreover, since the ambient space (M, g) is of constant curvature, $g(R^D(Y,Z)\xi,\eta) = 0$. Therefore, it follows from the Lemma 2 that $\mathcal{H}^D(\mathcal{L})$ along any leaf $\mathcal{L}$ is discrete.

COMMENT. Given a leaf $\mathcal{L}$ of a Riemannian foliation $\mathcal{V}$ with $x \in \mathcal{L}$, the foliation holonomy group along $\mathcal{L}$ is defined as the image of a homomorphism $\Psi : \pi_1(\mathcal{L}, x) \longrightarrow G$, where G is the group of germs of isometries of R^q preserving o. Let $J : G \longrightarrow O(q)$ be defined by

$$J(q) := \text{The Jacobian of } g \text{ at } o.$$

The image, denoted by $Hol(\mathcal{L})$, of the composition $J \circ \Psi$ is called the linear foliation holonomy group of $\mathcal{V}$ along $\mathcal{L}$ at x. If we apply the fact ([3]) that the linear foliation holonomy group of $\mathcal{V}$ along a leaf $\mathcal{L}$ can be identified with the Bott connection holonomy group restricted to loops in $\mathcal{L}$, it immediately follows from the definition of D that the normal holonomy group $H^D(\mathcal{L})$ of $\mathcal{V}$ along any leaf $\mathcal{L}$ with integrable horizontal bundle is discrete. Indeed, we have

PROPOSITION 3. *The normal holonomy group $\mathcal{H}^D(\mathcal{L})$ of a Riemannian foliation $\mathcal{V}$ coincides with its linear foliation holonomy group $Hol(\mathcal{L})$ if the horizontal bundle $\mathcal{H}$ of $\mathcal{V}$ is integrable.*

REMARK. The following example shows that the hypothesis that $\mathcal{V}$ is totally umbilic in the Main Theorem is necessary.

Let $G := SO(2) \times SO(3)$ and $\{e_0, e_1, e_2, e_3\}$ a basis of the Lie algebra $\underline{G}$ of all left-invariant vector fields on G satisfying $[e_1, e_2] = e_3 - e_1, [e_2, e_3] = e_1, [e_3, e_1] = e_0$, $[e_0, e_2] = e_0, [e_0, e_t] = 0 (t = 1, 3)$. Let $\underline{H}$ be a Lie subalgebra of $\underline{G}$ generated by $\{e_0, e_1\}$. Then we have a foliation $\mathcal{V}(\underline{H})$ on G. Let $T := (h^a_{ij})$ and $A := (A^a_{ib})$, where $i, j = 0, 1$ and $a, b = 2, 3$. If we take a left - invariant metric g on G so that $\{e_0, e_1, e_2, e_3\}$ is orthonormal, then

$$h^2_{00} = -1 = -h^2_{11},\ h^2_{01} = h^3_{00} = h^3_{11} = 0,\ h^3_{01} = \frac{1}{2},$$

and

$$A^2_{02} = A^2_{12} = A^2_{03} = A^3_{02} = A^3_{03} = A^3_{13} = 0, A^2_{13} = 1 = -A^3_{12}.$$

Moreover, by a direct computation, we have $g(R(e_0, e_1)e_2, e_3) = 0$, hence by (3) we obtain $g(R^D(e_0, e_1)e_2, e_3) = 1$. Note that $\mathcal{V}(\underline{H})$ on (G, g) is Riemannian, minimal and not totally geodesic. Therefore $\mathcal{V}(\underline{H})$ is not totally umbilic.

References

1. R.Bott, *Lectures on characteristic classes and foliations*, Lectures on Algebraic and Differential Topology, Lecture Notes in Math, Springer-Verlag and New York, 1973, pp. 1–94.
2. H. Kitahara and S. Yorozu, *On some differential geometric characterizations of a bundle - like metric*, Kodai Math. J. **2** (1979), 130–138.
3. A. Morgan, *Holonomy and metric properties of foliations in higher codimension*, Proc. Amer. Math. Soc. **58** (1976), 255–261.
4. C. Olmos, *The normal holonomy group*, Proc. Amer Math. Soc. **110** (1990), 813–818.
5. B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469.

DEPARTMENT OF MATHEMATICS GRADUATE SCHOOL, KANA- ZAWA UNIVERSITY KANAZAWA, ISHIKAWA 920, JAPAN.

DEPARTMENT OF MATHEMATICS TEACHERS COLLEGE, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU 702-701, KOREA.