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ON THE AFFINE WEYL GROUP OF TYPE $\tilde{A}_{n-1}$ II

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The affine Weyl group $\tilde{A}_{n-1}$ is an irreducible Coxeter group whose Coxeter graph is a polygon with n vertices [1,2,3]. A presentation for $\tilde{A}_{n-1}$ is

$$\begin{cases} y_1, y_2, \cdots y_n | y_i^2 = e, & 1 \le i \le n, \\ (y_i y_{i+1})^3 = e, & 1 \le i \le n-1, \\ (y_i y_j)^2 = e, & 1 \le i < j-1 < n \quad \text{and} \quad (i,j) \neq (1,n) \\ (y_1 y_n)^3 = e. & \end{cases}$$

In our paper [4] we showed that $\tilde{A}_{n-1}$ is a split extension of (n-1) copies of $\mathbf{Z}$ by the symmetric group S_n of degree n. We show in this paper how $\tilde{A}_{n-1}$ appears naturally as a subgroup of the natural wreath product $W = \mathbf{Z}S_n$.

A presentation for the group S_n is

$$\begin{cases} x_1, x_2, \cdots x_{n-1} | x_i^2 = e, & 1 \le i \le n-1, \\ (x_i x_{i+1})^3 = e, & 1 \le i \le n-2, \\ (x_i x_j)^2 = e, & 1 \le i < j-1 < n-1 \end{cases}$$

where x_i is the transposition $(i, i+1)$. The group S_n acts naturally on the set $X = \{1, 2, 3, \cdots n\}$. We define the base group B of the wreath product W to be the direct product of n copies of $\mathbf{Z}$. Thus

$$B = \langle a_1, a_2, \cdots, a_n | a_i a_j = a_j a_i, \quad 1 \le i < j \le n \rangle.$$

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We use the action of S_n on X to define a natural action of S_n on B as follows:

$$(1)\quad (a_1, a_2, \cdots, a_{i-1}, a_i, a_{i+1}, a_{i+2}, \cdots, a_n)^{X_i} = (a_1, a_2, \cdots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \cdots, a_n) \quad \text{for} \quad 1 \leq i \leq n-1.$$

The natural wreath product W is the split extension of B by S_n with action (1).

We consider the subgroup H of B containing all elements of exponent sum zero. Thus $H = \{a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n} | m \in Z, \sum_{i=1}^{n} m_i = 0\}$. Since H is a normal subgroup of B containing elements of the form $a_1 a_2^{-1}, a_1 a_3^{-1}, \cdots a_1 a_n^{-1}$, we find that $B/H = \langle a_1 | \rangle = \{a_1^k | k \in Z\}$.

We use the Reidemeister-Schreier process to find the following presentation for H:

$$H = \langle b_2, b_3, \cdots, b_n | b_i b_k = b_k b_i, \quad 2 \leq i < k \leq n \rangle \simeq \mathbf{Z}^{n-1}$$

where $b_i = a_i a_i^{-1}, 2 \leq i \leq n$.

H is closed under the action of S_n and we can find this action from (1) to be:

$$b_2^{x_1} = b_2^{-1}, b_i^{x_1} = b_i b_2^{-1}, 3 \leq i \leq n$$

$$(2)\qquad b_k^{x_i} = \begin{cases} b_{k+1} & \text{if} \quad i = k, \\ b_{k-1} & \text{if} \quad i = k-1 \\ b_k & \text{if} \quad i \neq k, i \neq k-1 \end{cases}$$

where $2 \leq i \leq n-1$ and $2 \leq k \leq n$.

We consider the group E which is the split extension of H by S_n with the action as in (2). Then E is obviously a subgroup of the wreath product $\mathbf{Z}\ S_n$. In our paper [4] we showed that E is isomorphic to $\tilde{A}_{n-1}$. Therefore, we have shown that $\tilde{A}_{n-1}$ is a subgroup of $\mathbf{Z}\ S_n$.

REMARK. We observe that the subgroup

$$K = \{a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n} | a_1 = a_2 = \cdots = a_n, m_i \in \mathbf{Z}\}$$

of B is isomorphic to $\mathbf{Z}$. We raise the question whether W/K is isomorphic to $\tilde{A}_{n-1}$ or not.

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