Bull. Korean Math. Soc. 30(1993), No. 1, pp. 65–70

ON UNBOUNDED SUBNOMAL OPERATORS

KYUNG HEE JIN

In this paper we will extend some notions of bounded linear operators to some unbounded linear operators. Let $\mathcal{H}$ be a complex separable Hilbert space and let $B(\mathcal{H})$ denote the algebra of bounded linear operators. A closed densely defind linear operator S in $\mathcal{H}$, with domain domS, is called ***subnormal*** if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator N in $\mathcal{K}$ (i.e.,$N^*N = NN^*$) such that domS $\subseteq$ domN and Sf = Nf for f $\in$ domS.

We will show that the Radjavi and Rosenthal theorem holds for some unbounded subnormal operators; if S_1 and S_2 are unbounded subnormal operators on $\mathcal{H}$ with $\mathrm{dom}S_1 = \mathrm{dom}S_1^*$ and $\mathrm{dom}S_2 = \mathrm{dom}S_2^*$ and $A \in B(\mathcal{H})$ is injective, has dense range and $S_1A \supseteq AS_2^*$, then S_1 and S_2 are normal and $S_1 \cong S_2^*$.

A linear operator $A : \mathcal{H} \longrightarrow \mathcal{K}$ is ***boundedly invertible*** if there is a bounded linear operator $B : \mathcal{K} \longrightarrow \mathcal{H}$ such that $AB = I$ and $BA \subseteq I$. The spectrum of A is the set $\sigma(A) = \{\lambda \in C : A - \lambda I$ is not boundedly invertible $\}$. The resovelent set $\rho(A)$ of A is the complement of $\sigma(A)$. Unlike the case of the bounded linear operators, the spectrum of unbounded linear operator may be an empty set or the whole complex plane. For this reason, we will put some conditions on the spectrum, which are not necessary for bounded operator theory, in the following theorems.

LEMMA 1. *Let A and B be boundedly invertible in $\mathcal{H}$ and $C \in B(\mathcal{H})$. If $CA \subseteq BC$, then $B^{-1}C = CA^{-1}$.*

Proof. Given h in $\mathcal{H}$, $A^{-1}h \in domA$, $B^{-1}h \in domB$ and so $CA^{-1}h$ is in $domB$. Since $domB = ranB^{-1}$, $CA^{-1}h = B^{-1}g$ for some $g \in \mathcal{H}$ and $Ch = CAA^{-1}h = BCA^{-1}h = BB^{-1}g = g$.

Received January 30, 1992. Revised July 25, 1992.
This paper was partially supported by KOSEF-TGRC .

Hence $B^{-1}Ch = B^{-1}g = CA^{-1}h$ for all $h \in \mathcal{H}$ and so $B^{-1}C = CA^{-1}$.

FUGLEDE - PUTNAM THEOREM. *If N and M are normal operators in $\mathcal{H}$ and A is a bounded linear operator on $\mathcal{H}$ such that $MA \supseteq AN$, then $M^*A \supseteq AN^*$.*

Proof. See [6].

For the definition of quasi-similarity between unbounded operators and the proof of the following theorem, see [4].

THEOREM 2. *Let N and M be normal operators in $\mathcal{H}$. If N and M are quasi-similar, then N and M are unitarily equivalent.*

A closed densely defined operator $T : \mathcal{H} \longrightarrow \mathcal{H}$ is called ***hyponormal*** if $domT \subseteq domT^*$ and $\|Th\| \geq \|T^*h\|$ for h$\in$ $domT$. Every subnormal operator is hyponormal.

In [4], Ota and Schmudgen showed that for a hyponormal operator A and a closed densely defined operator B in $\mathcal{H}$, if there exists a bounded linear operator W with dense range such that $WB \subseteq AW$, then the spectrum of A is contained in that of B. In the following theorems we will use this.

THEOREM 3. *Let T be a hyponormal operator and let N be a normal operator in $\mathcal{H}$ such that the spectrum of T is not the whole complex plane. If $TW \supseteq WN$ where $W \in B(\mathcal{H})$ has dense range, then T is normal.*

Proof. For any $\lambda \in C$, $T - \lambda I$ is hyponormal and $N - \lambda I$ is normal and $(T - \lambda I)W \supseteq W(N - \lambda I)$. Since the resolvent of N is not empty and it is contained in the resolvent of T without loss of generality we may assume that 0 is in the resolvent of T. That is, assume that both T and N are boundedly invertible. It comes from Lemma 1 that $T^{-1}W = WN^{-1}$. Since T^{-1} is bounded hyponormal and N^{-1} is bounded normal, it follows from the Theorem 1 in [7] that T^{-1} is normal. Therefore T is normal.

The next corollary follows from the Fuglede-Putnam theorem, Theorem 2 and Theorem 3.

COROLLARY 4. *Let T be a hyponormal operator and let N be a normal operator in $\mathcal{H}$ with $\rho(N) \neq \emptyset$. If $W \in B(\mathcal{H})$ is injective and has dense range and $TW \supseteq WN$, then T and N are unitarily equivalent.*

Since a subnormal operator is hyponormal, this corollary also holds for subnormal operators.

THEOREM 5. *Let S_1 and S_2 be subnormal operators in $\mathcal{H}$ such that $\rho(S_2)$ is nonempty. If there is a bounded invertible operator R on $\mathcal{H}$ such that $S_1R \supseteq RS_2$ and $S_1^*R \supseteq RS_2^*$, then S_1 and S_2 are unitarily equivalent.*

Proof. Without losing of generasity we can assume that both S_1 and S_2 are boundedly invertible. Thus by the Lemma 1 $S_1^{-1}R = RS_2^{-1}$ and $(S_1^{-1})^*R = R(S_2^{-1})^*$. Since S_1^{-1} and S_2^{-1} are bounded subnormal, by the fact for bounded subnormal operators (see [1]), we have the equivalence between S_1^{-1} and S_2^{-1}. Therefore S_1 and S_2 are unitarily equivalent.

In [4], Ota and Schmudgen showed that quasi-similar hyponormal operators have equal spectra. Thus we have the following corollary.

COROLLARY 6. *Under the same conditions as in Theorem 5, $\sigma(S_1) = \sigma(S_2)$.*

It is easy to check that the domain of a subnormal operator is contained in the domain of its adjoint operator. As we see in the closed symmetric (not self-adjoint) operator, the domain of a subnormal operator need not to be the same as that of its adjoint. (For the proof that the closed symmetric operator is subnormal, see [1] Example 2.10 or [8] Proposition 1.) But there is also subnormal operator such that the domain is equal to that of its adjoint. For example, if S is a subnormal operator with bounded real part, then the domain of S is equal to the domain of its adjoint. For details see [2].

LEMMA 7. *Let S be a subnormal operator in $\mathcal{H}$ with $domS = domS^*$. If N is any normal extension of S in $\mathcal{K}$, then $domN = (domN \cap \mathcal{H}\ ) \oplus (domN \cap \mathcal{H}^\perp)$.*

Proof. Let $f \in domN$. Since $domS = domS^*$, $\mathrm{P}f \in domS$ where P is the projection of $\mathcal{K}$ onto $\mathcal{H}$. Since $\mathrm{P}^\perp f = f - \mathrm{P}f$ is in $domN \cap \mathcal{H}^\perp$

$f = \mathrm{P}f + \mathrm{P}^{\perp}f$ is in $(domN \cap \mathcal{H}) \oplus (domN \cap \mathcal{H}^{\perp})$ and so $domN \subseteq (domN \cap \mathcal{H}) \oplus (domN \cap \mathcal{H}^{\perp})$.

It is easy to check the other inclusion.

Let A be a closed densely defined operator in $\mathcal{H}$. A closed subspace $\mathcal{H}_0 \subseteq \mathcal{H}$ *reduces* A if (i) $A(\mathcal{H}_0 \cap domA) \subseteq \mathcal{H}_0$ and (ii) $A^*(\mathcal{H}_0 \cap domA) \subseteq \mathcal{H}_0$.

If N is a normal operator in $\mathcal{H}$ and $\mathcal{H}_0$ reduces N, then $N_0 = N|_{\mathcal{H}_0}$ is normal.

From the above Lemma 7 , we know that if S is a subnormal with $\mathrm{dom}S = \mathrm{dom}S^*$, then any normal extension of S can be represented as an operator matrix.

The following theorem is an unbounded version of the Radjavi-Rosenthal theorem ([5]) on bounded subnormal operators.

THEOREM 8. *Let S_1 and S_2 be subnomal operators on $\mathcal{H}$ with $domS_1 = domS_1^*$ and $domS_2 = domS_2^*$. If $A \in B(\mathcal{H})$ is positive and injective and $S_1 A \supseteq AS_2^*$, then S_1 and S_2 are normal and $S_1 \cong S_2^*$. Moreover, if $\sigma(S_1)$ is not the whole complex plane, then $S_1 = S_2^*$.*

Proof. Let N_1 and N_2 be normal extentions of S_1 and S_2, respectively, acting on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^{\perp}$. Let for i = 1,2 $domD_i = \{P^{\perp}f : f \in domN_i\}$ where $P^{\perp}$ is the projection of $\mathcal{K}$ onto $O \oplus H^{\perp}$. Then $domN_i = domS_i \oplus domD_i$ for i = 1,2.

Let

$$N_1 = \begin{pmatrix} S_1 & X_1 \\ 0 & T_1^* \end{pmatrix}, N_2 = \begin{pmatrix} S_2 & X_2 \\ 0 & T_2^* \end{pmatrix}$$

be the matrices relative to $\mathcal{H} \oplus \mathcal{H}^{\perp}$.

Also let $A_0 = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ relative to $\mathcal{H} \oplus \mathcal{H}^{\perp}$. Since $S_1 A \supseteq AS_2^*$, $N_1 A_0 \supseteq A_0 N_2^*$. So it follows from the Fuglede - Putnam's Theorem that $N_1^* A_0 \supseteq A_0 N_2$.

Therefore

$$N_2(KerA_0 \cap domN_2) \subseteq KerA_0$$

and

$$N_2^*(KerA_0 \cap domN_2) \subseteq KerA_0$$

Since A is injective, it follows that $KerA_0$ reduces N_2 and hence $X_2 = 0$. So S_2 is normal.

On the other hand, by taking adjoint on both sides of $S_1 A \supseteq AS_2{}^*$ we have $S_2 A \supseteq AS_1{}^*$. Now it comes from the preceding argument that S_1 is also normal. Since S_1 and S_2 are normal, it follows from the Fugled-Putnam's theorem that $S_2{}^* A \supseteq AS_1$. Therefore S_1 and S_2^* are quasi-similar and they are unitarily equivalent.

Now assume that $\sigma(S_1)$ is not the whole complex plane. Note that the relations $S_1 A \supseteq AS_2^*$ and $S_2{}^* A \supseteq AS_1$ imply that $\sigma(S_1) = \sigma(S_2)$ and $S_1 A^2 \supseteq A^2 S_1$ and ,using the spectral theory, we have $S_1 A \supseteq AS_1$. Let $z_0 \notin \sigma(S_1)$. Then the inclusions

$$(z_0 I - S_1)A \supseteq A(z_0 I - S_1) and (z_0 I - S_1)A \supseteq A(z_0 I - S_2^*)$$

imply that

$$A(z_0 I - S_2^*)^{-1} = (z_0 I - S_1)^{-1} A = A(z_0 I - S_1)^{-1}.$$

Hence $(z_0 I - S_1)^{-1} = (z_0 - S_2{}^*)^{-1}$ and so $S_1 = S_2{}^*$.

COROLLARY 9. *Let S_1 and S_2 be subnormal on $\mathcal{H}$ with $domS_1 = domS_1^*$ and $domS_2 = domS_2^*$. If $A \in B(\mathcal{H})$ is injective, has dense range and $S_1 A \supseteq AS_2^*$, then S_1 and S_2 are normal and S_1 and S_2^* are unitarily equivalent.*

Proof. Let $A = UP$ be the polar decomposition of A where U is unitary and P is positive. Since

$$U^* S_1 UP = U^* S_1 A \supseteq U^* AS_2^* = PS_2^*$$

and $U^* S_1 U$ is subnormal, it follows from theorem 8 that US_1U and S_2 are normal and US_1U and S_2^* are unitarily equivalent. Therefore S_1 and S_2 are normal and S_1 and S_2^* are unitarily equivalent.

ACKNOWLEDGEMENT.

I am very grateful to Professor Hong Oh Kim for his hospitality during preparation of this paper at KAIST. And I am also indebted to the referee for improving the first draft of this paper.

References

1. J.B. Conway,, *A course in Functional Analysis*, Springer-Verlag,N.Y.Inc.1985.
2. Kyung Hee Jin, *On Unbounded Subnormal Operators*, Indiana Univ. Thesis,(1989).
3. G. McDonald and C. Sundberg, *On the Spectra of Unbounded Subnormal Operators*, Can. J. Math. **38** (1986), 1153– 1148.
4. S. Ota and K. Schmudgen, *On Some Classes of Unbounded Operators*, Int. Eq. and Oper. Theory, Vol. 12(1989), 211–226.
5. H.Radjavi and P. Rosenthal, *On Roots of Normal Operators*, J. Math. Anal. Appl. **34** (1971), 653 - 664.
6. C.R. Putnam, *On Normal Operators in Hilbert Space*, Amer. J. Math. **73** (1951), 357 - 362.
7. J.G.Stampfli and B.L.Wadhwa, *An Asymmetric Putnam- Fuglede Theorem for Dominant Operators*, Indiana Univ. Math. J. 25 no.4(1976).
8. Jan Stochel and F.H. Szafraniec, *A few assorted questions about Unbounded Subnormal Operators*, UWV Iagel Acta Math..

Department of Mathematics, Kangweon National Univerty, Chuncheon 200-701,Korea