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SOME REMARKS ON PRIMAL IDEALS

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1. Introduction

Every ring considered in this paper will be assumed to be commutative and have a unit element. An ideal A of a ring R will be called primal if the elements of R which are zero divisors modulo A, form an ideal of R, say P. If A is a primal ideal of R, P is called the adjoint ideal of A. The adjoint ideal of a primal ideal is prime [2]. The definition of primal ideals may also be formulated as follows: An ideal A of a ring R is primal if in the residue class ring R/A the zero divisors form an ideal of R/A. If Q is a primary idel of a ring R then every zero divisor of R/Q is nilpotent; therefore, Q is a primal ideal of R. That a primal ideal need not be primary, is shown by an example in [2].

Let $R[X]$ and $R[[X]]$ denote the polynomial ring and formal power series ring in an indeterminate X over a ring R, respectively. Let S be a multiplicative system in a ring R and $S^{-1}R$ the quotient ring of R. Let Q be a P-primary ideal of a ring R. Then $Q[X]$ is a $P[X]$-primary ideal of $R[X]$, and $S^{-1}Q$ is a $S^{-1}P$-primary ideal of a ring $S^{-1}R$ if $S \cap P = \emptyset$, and $Q[[X]]$ is a $P[[X]]$-primary ideal of $R[[X]]$ if R is Noetherian [1]. We search for analogous results when primary ideals are replaced with primal ideals. To show an ideal A of a ring R to be primal, it sufficies to show that $a-b$ is a zero divisor modulo A whenever a and b are zero divisors modulo A.

DEFINITION. *An ideal A of a R is irreducible if A can not be expressed as a finite intersection of proper divisors of A*

A primal ideal may not be irreducible but every irreducible ideal is primal [2]. Without using this result, directly we can prove that if A is an irreducible ideal of ring R, $A[X]$ is a primal ideal of $R[X]$. (First part of Proposition 1).

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PROPOSIRION 1. *Let A be an irreducible ideal of a ring R. Then $A[X]$ is a primal ideal of $R[X]$. Furthermore, if P is the adjoint ideal of A considered as a primal ideal of R, then $P[X]$ is the adjoint ideal of $A[X]$.*

Proof. Let A be an irreducible ideal of R. For each $f(X) = \sum_{i=0}^{m} a_i X^i \in R[X]$, we define $\bar{f}(X)$ to be $\sum_{i=0}^{m} \bar{a}_i X^i$ where $\bar{a}_i = a_i + A \in R/A$ for each $i = 0, \cdots, m$. Then $\bar{f}(X) \in R/A[X]$. Since the mapping $\phi : R[X]/A[X] \longrightarrow R/A[X]$ defined by $\phi(f(X) + A[X]) = \bar{f}(X)$ is an isomorphism and onto, we see that $f(X)$ is a zero divisor modulo $A[X]$ if and only if $\bar{f}(X)$ is a zero divisor in $R/A[X]$. Let $g(X) = \sum_{i=0}^{n} b_i X^i$ and $h(X) = \sum_{i=0}^{p} c_i X^i$ be zero divisors modulo $A[X]$ Then $\bar{g}(X)$ and $\bar{h}(X)$ are zero divisors in $R/A[X]$.

By McCoy's theorem, there exist nonzero elements $\bar{r} = r + A$ and $\bar{s} = s + A$ in R/A such that $\bar{r}\bar{g}(X) = \bar{0}$ and $\bar{s}\bar{h}(X) = \bar{0}$. Clearly, $(r) + A$ and $(s) + A$ are proper divisors of A; therefore, $[(r) + A] \cap [(s) + A]$ is a proper divisor of A. So there exists $v \in [(r) + A \cap [(s) + A]$ such that $v \notin A$. Then $v = rt_1 + a_1 = st_2 + a_2$ for some $t_1, t_2 \in R$ and $a_1, a_2 \in A$. Note that $rt_1, st_2 \notin A$ and $\bar{v} = \overline{rt_1} = \overline{st_2} \neq \bar{0}$. But $\bar{v}(\bar{g}(X) - \bar{h}(X)) = \overline{rt_1}\bar{g}(X) - \overline{st_2}\bar{h}(X) = \bar{0}$; therefore, $\bar{g}(X) - \bar{h}(X)$ is a zero divisor in $R/A[X]$, so $g(X) - h(X)$ is a zero divisor modulo $A[X]$.

Thus $A[X]$ is primal. Let P be an adjoint ideal of A. We show that $P[X]$ is the adjoint ideal of $A[X]$.

Let $f(X) = \sum_{i=0}^{n} a_i X^i$ be a zero divisor modulo $A[X]$. Then $\bar{f}(X) = \sum_{i=0}^{n} \bar{a}_i X^i$ is a zero divisor in $R/A[X]$ so there exists $\bar{r} \in R/A$, $\bar{r} \neq \bar{0}$ such that $\bar{r}\bar{f}(X) = \bar{0}$. Then all $\bar{a}_i$ are zero divisors in R/A and all a_i are zero divisors modulo A, so all a_i are in P. So $f(X) \in P[X]$, which implies that all zero divisors modulo $A[X]$ are contained in $P[X]$. Let $q(X) = \sum_{i=0}^{n} d_i X^i \in P[X]$. We show that $q(X)$ is a zero divisor modula $A[X]$. If $q(X) \in A[X]$, then clearly $q(X)$ is a zero divisor modulo $A[X]$.

So we assume $q(X) \notin A[X]$. Suppose that $d_{i_1}, d_{i_2}, \cdots, d_{i_s} \notin A$ and all other d_i are in A. Then there exist $t_1, t_2, \cdots, t_s \in P - A$ such that $t_1 d_{i_1}, t_2 d_{i_2}, \cdots, t_s d_{i_s} \in A$. Let $D = [(t_1) + A] \cap [(t_2) + A] \cap \cdots \cap [(t_s) + A]$, then D is a proper divisor of A. Since A is irreducible, there exists d in D such that $d \notin A$.

Then $d = r_1 t_1 + a_1 = r_2 t_2 + a_2 = \cdots = r_s t_s + a_s$ for some $r_1, r_2, \cdots, r_s$

$\in R$ and $a_1, a_2, \cdots, a_s \in A$. Since $t_1 d_{i_1}, \cdots, t_s d_{i_s} \in A$, $d \cdot q(X) = d \cdot \sum_{i=0}^{n} d_i X^i \in A[X]$. Note that $d \notin A[X]$. Thus $q(X)$ is a zero divisor modulo $A[X]$. We showed that every element of $P[X]$ is a zero divisor modulo $A[X]$. Thus $P[X]$ is the disjoint ideal of $A[X]$.

2. Main Results

Natually, the following question arises: If A is a primal ideal of a ring R with the adjoint prime ideal P, is $A[X]$ a primal ideal of $R[X]$ with the adjoint prime ideal of $P[X]$? In Theorem 1 we will see that the answer of this question is not affirmative.

A Noetherian ring has the property that annihilator of each ideal consisting entirely of zero divisors is nonzero [4; p.56]. Huckaba [3] abstracted this to aribitrary ring as following definition.

DEFINITION. *A ring satisfies Property (*) if each finitely generated ideal consisting entirely of zero divisors has nonzero annihilator.*

Every polynomial ring $R[X]$ satisfies Property (*) and every zero-dimensional ring satisfies Property (*)[3; p.7,9].

THEOREM 1. *Let A be a primal ideal of a ring R with the adjoint prime ideal P. Then R/A satisfies property (*) if and only if $A[X]$ is a prime ideal of $R[X]$ with the adjoint prime ideal $P[X]$.*

Proof. Suppose that R/A satisfies Property (*). Let $F(X) = \sum_{i=0}^{m} a_i X^i$ and $g(X) = \sum_{i=0}^{n} b_i X^i$ be zero divisors modulo $A[X]$. Then $\bar{f}(x) = \bar{g}(x)$ are zero divisors in $R/A[X]$, so $\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_m, \bar{b}_1, \bar{b}_2, \cdots, \bar{b}_n$ are zero divisors in R/A. Then $a_1, a_2, \cdots, a_m$, $b_1, b_2, \cdots, b_n$ are zero divisors modulo A. Let $B = (a_1, a_2, \cdots, a_m, b_1, b_2, \cdots, b_n)$. Then $B \subseteq P$ since P is an ideal and consists of all zero divisors modulo A. Then B consists entirely of zero divisors modulo A, so the ideal $\bar{B} = (\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_m, \bar{b}_1, \bar{b}_2, \cdots, \bar{b}_n)$ consists entirely of zero divisors of R/A. Since R/A satisfies Property (*), there exists $\bar{r} \in R/A$, $\bar{r} \neq \bar{o}$ such that $\bar{r} \cdot \bar{B} = (\bar{o})$. Then $\bar{r}(\bar{f}(X) - \bar{g}(X)) = \bar{o}$, so $\bar{f}(X) - \bar{g}(X)$ is a zero divisor in $R/A[X]$ therefore, $f(X) - g(X)$ is a zero divisor modulo $A[X]$. Thus $A[X]$ is a prime ideal of $R[X]$.

Next, we show that $P[X]$ is the adjoint ideal of $A[X]$. That all zero divisors modulo $A[X]$ are contained in $P[X]$, can be proved in the same way as in the proof of Proposition 1, so we omit its proof. Let $q(x) = \sum_{i=0}^{n} d_i X^i \in P[X]$. We will show that $q(X)$ is a zero divisor modulo $A[X]$. Let $D = (d_1, \cdots, d_n)$, then $D \subseteq P$ and D consists entirely of zero divisors modula A. Then the ideal $\bar{D} = (\bar{d}_1, \cdots, \bar{d}_n)$ consists entirely of zero divisors in R/A where $\bar{d}_i = d_i + A \in R/A$ for each i. Since R/A satisfies Property (*), there exists $\bar{r}$ in R/A, $\bar{r} \neq \bar{0}$ such that $\bar{r}\bar{D} = (\bar{0})$. Hence $\bar{q}(x) = \sum_{i=0}^{n} \bar{d}_i X_i$ is a zero divisor in $R/A[X]$; Therefore, $q(x)$ is a zero divisor modulo $A[X]$. Thus $P[X]$ is an adjoint ideal of $A[X]$.

Conversely, suppose that $A[X]$ is a primal ideal of $R[X]$ with its adjoint ideal $P[X]$. Let $\bar{U} = (\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n)$ be an ideals of R/A consisting entirely of zero divisors of R/A. Then $u_1, u_2, \cdots, u_n$ are zero divisors modulo A. So $\sum_{i=0}^{n} u_i X^i \in P[X]$. Then $\sum_{i=0}^{n} u_i X^i$ is a zero divisor modulo $A[X]$. Hence $\sum_{i=0}^{n} \bar{u}_i X^i$ is a zero divisor of $R/A[X]$. Then there exists $\bar{v} \in R/A$, $\bar{v} \neq \bar{0}$ such that $\bar{v}\bar{u}_i = \bar{0}$ for each $i = 0, 1, \cdots, n$. So $\bar{v} \cdot \bar{U} = \bar{0}$. Thus the ring R/A satisfies Property (*).

COROLLARY 1. *If A is an irreducible ideal of a ring R, then R/A satisfies Property (*).*

Proof. Let A be an irreducible ideal of a ring R and P its adjoint ideal. By Proposition 1, $A[X]$ is a primal ideal of $R[X]$ with the adjoint ideal $P[X]$. Then by Theorem 1, R/A satisfies Property (*).

COROLLARY 2. *Let A be a primal ideal of a ring R with the adjoint ideal P. Then if R/A satisfies Property (*), $A[X_1, \cdots, X_n]$ is primal ideal of $R[X_1, \cdots, X_n]$ with the adjoint ideal $P[X_1, \cdots, X_n]$.*

Proof. Let A be a primal ideal of a ring R with the adjoint ideal P. Assume that R/A satisfies Property (*). Then $A[X_1]$ is a primal ideal of $R[X_1]$ with the adjoint ideal $P[X_1]$ (by Theorem 1). Since the polynomial ring $R/A[X_1]$ satisfies Property (*)[3;p.7] and $R/A[X_1] \simeq R[X_1]/A[X_1]$, it follows that $R[X_1]/A[X_1]$ satisfies (*). Then by Theorem 1, $A[X_1, X_2]$ is a primal ideal of $R[X_1, X_2]$ with the adjoint ideal $P[X_1, X_2]$.

THEOREM 2. *Let A be an ideal of a ring R. Then if $A[X]$ (resp. $A[[X]]$) is a primal ideal of $R[X]$ (resp. $R[[X]]$), then A is primal.*

Proof. Let a_1, and a_2 be elements of R which are zero divisors modulo A. Then there exist b_1 and b_2 in R such that $b_1, b_2 \notin A, a_1b_1 \in A$ and $a_2b_2 \in A$. Then $b_1, b_2 \notin A[X]$, $a_1b_1 \in A[X]$ and $a_2b_2 \in A[X]$, so a_1 and a_2 are zero divisors modulo $A[X]$. Since $A[X]$ is a primal ideal of $R[X]$, there exists $g(X) = \sum_{i=0}^{n} c_iX^i$ in $R[X]$ such that $g(X) \notin A[X]$ and $g(X)(a_1 - a_2) \in A[X]$. Then $c_j(a_1 - a_2) \in A$ for some $c_j \notin A$; therefore, $a_1 - a_2$ is a zero divisor modulo A, and A is a primal ideal of R. Similarly, we can prove the Theorem when $A[X]$ and $R[X]$ are replaced by $A[[X]]$ and $R[[X]]$, respectively.

THEOREM 3. *Let A is a primal ideal of R with the adjoint ideal P and let S be a multiplicative system in R such that $S \cap P$ is empty. Then $S^{-1}A$ is a primal ideal of $S^{-1}R$ with the adjoint ideal $S^{-1}P$.*

Proof. We show that if a/t is a zero divisor modulo $S^{-1}A$, then a is a divisor module A. Let a/t be a zero divisor modulo $S^{-1}A$, then there exist $b/s \in S^{-1}R$ such that $b/s \notin S^{-1}A$ and $(a/t)\cdot(b/s) \in S^{-1}A$. Then there exists v in S such that $abv \in A$. Clearly, $bv \notin A$, for otherwise $b/s \in S^{-1}A$ which violates $b/s \notin S^{-1}A$. Hence a is a zero divisor modulo A. To prove $S^{-1}A$ to be primal, let a_1/t_1 and a_2/t_2 be zero divisors modulo $S^{-1}A$. Then a_1 and a_2 are zero divisors modulo A. Since A is a primal ideal with adjoint ideal P, $a_1t_2 - a_2t_1 \in P$. Then there exists r in $P - A$ such that $(a_1t_2 - a_2t)r \in A$.

Then $(a_1t_2 - a_2t_1)r/t_1t_2u = (a_1/t_1 - a_2/t_2)(r/u) \in S^{-1}A$ for any $u \in S$. Claim $r/u \notin S^{-1}A$. For suppose $r/u \in S^{-1}A$, then there exists v in S such that $vr \in A$. Since $r \in P - A$, v is a zero divisor modulo A so $v \in P$. Then $v \in S \cap P$ which violates our assumption $S \cap P = \emptyset$. Hence $r/u \notin S^{-1}A$ and $a_1/t_1 - a_2/t_2$ is a zero divisor modulo $S^{-1}A$. Therefore, $S^{-1}A$ is a primal ideal of $S^{-1}R$. Next we show that $S^{-1}P$ is the adjoint ideal of $S^{-1}A$.

Let a/t be a zero divisor modulo $S^{-1}A$, then a is a zero divisor modulo A; therefore, $a \in P$ and $a/t \in S^{-1}P$. This shows that every zero divisor modulo $S^{-1}A$ is contained in $S^{-1}P$. Let $b/s \in S^{-1}P$, then $bd \in P$ for some $d \in S$. Since P is a prime ideal and $d \notin P$, it follows that $b \in P$ and b is a zero divisor modulo A. Then there exists c in $P - A$ such that $bc \in A$. Then $(b/s)(c/t) \in S^{-1}A$ for any $t \in S$. Claim $c/t \notin S^{-1}A$. For suppose $c/t \in S^{-1}A$, then $cv \in A$ for some $v \in S$.

Note that v is a regular element modulo A, so $c \in A$. But $c \in P - A$ which leads a contradiction. So $c/t \notin S^{-1}A$ and b/s is a zero divisor modulo $S^{-1}A$. This shows that every element of $S^{-1}P$ is a zero divisor modulo $S^{-1}A$. Thus we can conclude that $S^{-1}P$ is the set of all zero divisors modulo $S^{-1}A$; therefore, $S^{-1}P$ is the adjoint ideal of $S^{-1}A$.

Let A be an ideal of a ring R and S a multiplicative system in R. Consider the mapping $\phi; R \longrightarrow S^{-1}R$ defined by $\phi(a) = as/s$ for $s \in S$. Then ϕ is a ring homomorphism. Let $S^{-1}A \cap R$ denote the complete inverse image of $S^{-1}A$ under ϕ. Then $S^{-1}A \cap R$ is the contraction of $S^{-1}A$ to R.

THEOREM 4. *Let A be an ideal of a ring R and S a multiplicative system in R such that $S \cap A$ is empty. Then if $S^{-1}A$ is a primal ideal of $S^{-1}R$, then $S^{-1}A \cap R$ is a primal ideal of R.*

Proof. Let $A_S = \{x \in R | sx \in A \text{ for some } s \in S\}$. Then it follows that $S^{-1} \cap R = A_S$ [5;p.69]. Let a be a zero divisor modulo A_S. Then there exists b in R such that $b \notin A_S$ and $ab \in A_S$. Then $sab \in A$ for some $s \in S$. Then $ab/s_1s_2 \subset S^{-1}A$ for any $s_1, s_2 \in S$. Claim $b/s_2 \notin S^{-1}A$. For suppose $b/s_2 \in S^{-1}A$. Then $tb \in A$ for some $t \in S$, hence $b \in A_S$ which violates $b \notin A_S$. So $b/s_2 \notin S^{-1}A$; therefore, a/s_1 is a zero divisor modulo $S^{-1}A$ for any $s \in S$. Let a_1 and a_2 be elements of R which are zero divisors modulo A_S. Then t_1a_1 and t_2a_2 are zero divisors modulo A_S for any $t_1, t_2 \in S$. Then t_1a_1/t_1 and t_2a_2/t_2 are zero divisors modulo $S^{-1}A$. Since $S^{-1}A$ is a primal idel, there exists a_3/t_3 in $S^{-1}R$ such that $a_3/t_3 \notin S^{-1}A$ and $(a_1t_1/t_1 - a_2t_2/t_2)(a_3/t_3) \in S^{-1}A$. Then there exists t_4 in S such that $(a_1t_1t_2 - a_2t_1t_2)a_3t_4 \in A$. Therefore, $(a_1 - a_2)a_3 \in A_S$. Since $a_3/t_3 \notin S^{-1}A$, we see that $a_3t \notin A$ for any $t \in S$. Then $a_3 \notin A_S$ so $a_1 - a_2$ is a zero divisor modulo A_S. Thus $S^{-1}A \cap R (= A_s)$ is a primal ideal of R.

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