Bull. Korean Math. Soc. 30(1993), No. 1, pp. 91–97

ON THE EXISTENCE OF A UNIQUE INVARIANT PROBABILITY FOR A CLASS OF MARKOV PROCESSES

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1. Introduction

Let $(S, \mathcal{S})$ be a measurable space, Γ a set of $\mathcal{S}$-measurable mappings on S into S. Endow Γ with a σ-field $\mathcal{J}$ such that $(\gamma, x) \longrightarrow \gamma(x)$ is measurable from $(\Gamma \times S, \mathcal{J} \otimes \mathcal{S})$ to $(S, \mathcal{S})$, and let P be a probability measure on $\mathcal{J}$.

Each P and initial distribution, the distribution of X_0, determines a Markov process $\{X_n : n \geq 0\}$ with state space S and one-step transition probability $P(x, B)$ on $(S, \mathcal{S})$ defined by

$$P(x, B) = P(\{\gamma \in \Gamma : \gamma(x) \in B\}), \quad x \in S, B \in \mathcal{S},$$

where for fixed $B \in \mathcal{S}$, $P(\cdot, B)$ is a measurable function on S, and for fixed $x \in S, P(x, \cdot)$ is a probability measure on $\mathcal{S}$.

Markov process $\{X_n\}$ which is generated by the above manner is not in general irreducible.

Denote P^n by the joint distribution of $\alpha_1, \alpha_2, \cdots, \alpha_n$ where $\alpha_1, \alpha_2, \cdots$

is a sequence of independent identically distributed random maps on some probability space taking values in Γ with common distribution P, i.e., $P^n = P \times P \times \cdots \times P$ on $(\Gamma^n, \mathcal{S}^{\otimes n})$.

We shall write $P^{(n)}(x, dy)$ for the n-step transition probability, with $P^{(1)}(x, dy) = P(x, dy)$. Then $P^{(n)}(x, dy)$ is the distribution of $\alpha_n \alpha_{n-1} \cdots \alpha_1 x$.

Received April 24, 1992.

This paper was supported by Nondirected Research Fund, Korea Research Foundation, 1990

Define the adjoint operator T^* on $\mathcal{P}(S)$ of all probability measures on $\mathcal{S}$ by

$$T^*\mu(B) = \int_S P(x,B)\mu(dx), \quad \text{for every} \quad B \in \mathcal{S},$$

and define T^{*n} on $\mathcal{P}(S)$ by

$$T^{*n}\mu(B) = \int_S P^{(n)}(x,B)\mu(dx), \quad \text{for every} \quad B \in \mathcal{S}, n \geq 1.$$

$T^{*1} = T^*$.

Any element π in $\mathcal{P}(S)$ is called an invariant probability for the transition probability $P(x,dy)$ if $T^*\pi = \pi$.

In this article, we consider the case that S is a topologically complete subspace of R^k, and that Γ is a set of monotone functions on S into S.

It is obtained the sufficient condition for the existence of a unique invariant probability to which $P^{(n)}(x,dy)$ converges exponentially fast in a metric stronger than the Kolmogorov's distance. This extends the earlier results of Bhattacharya and Lee (1988) who considered the case Γ a set of nondecreasing functions.

II. Existence of a unique invariant probability

Let $S \subset R^k$ be topologically complete in its relativized Euclidean topology and let $\mathcal{B}(S)$ be the Boral σ-filed of S. For Γ we take the set of all continuous monotone (decreasing or increasing) functions $\gamma = (\gamma^{(1)}, \gamma^{(2)}, \cdots, \gamma^{(k)})$ on S into itself. In other words $\gamma^{(i)}(x^{(1)}, x^{(2)}, \cdots, x^{(k)})$, $1 \leq i \leq k$, is monotone decreasing in each coordinate $x^{(1)}, x^{(2)}, \cdots, x^{(k)}$ or monotone increasing in each coordinate. We shall often write γx for $\gamma(x)$.

Let $\mathcal{J}$ be a σ-field on Γ such that the map $(\gamma, x) \longrightarrow \gamma x$ is measurable on $(\Gamma \times S, \mathcal{J} \otimes \mathcal{B}(S))$ into $(S, \mathcal{B}(S))$.

Let $\mathcal{A}$ be the class of all sets $A \subset S$ of the form

$$A = \{y \in S : \gamma_n\gamma_{n-1}\cdots\gamma_1(y) \leq x\}$$

where $(\gamma_1, \gamma_2, \cdots, \gamma_n) \in \Gamma^n$ $(n \geq 1)$ and $x \in R^k$.

Let $\mathcal{P}(S)$ be the set of all probability measures on $(S, \mathcal{B}(S))$.
Define the distance d on $\mathcal{P}(S)$ by

$$(2.1) \qquad d(\mu,\nu) = \sup_{A\in\mathcal{A}}\{|\mu(A)-\nu(A)|\}, \quad \mu,\nu\in\mathcal{P}(S).$$

The topology on $\mathcal{P}(S)$ defined by d is stronger than the weak-star topology.

LEMMA 2.1. *The space $\mathcal{P}(S)$ is complete under the distance d defined by (2.1).*

Proof. In [1] Bhattacharya and Lee shows that when Γ is a set of continuous nondecreasing function. The proof of this lemma goes virtually the same line by line as lemma 2.2 [1], and we omit it.

Now we make the assumption on P;
There exists $x_0 \in S$ and a positive integer m such that

$$(2.2) \qquad P^m(\Gamma_1) > 0 \quad \text{and} \quad P^m(\Gamma_2) > 0$$

where

$$\begin{cases} \Gamma_1 = \{(\gamma_1,\gamma_2,\cdots,\gamma_m)\in\Gamma^m : \gamma_m\cdots\gamma_1(y)\le x_0 \quad \forall y\} \\ \Gamma_2 = \{(\gamma_1,\gamma_2,\cdots,\gamma_m)\in\Gamma^m : \gamma_m\cdots\gamma_1(y)\ge x_0 \quad \forall y\}. \end{cases}$$

Before stating the main theorem, we prove the following lemmas.

LEMMA 2.2. *T^* is a contraction on $\mathcal{P}(S)$.*

Proof. If γ is a continuous monotone function on S into S, then so is γ^{-1}, and hence $\gamma^{-1}\mathcal{A}\subset\mathcal{A}$, $\forall\gamma\in\Gamma$.
For $\mu,\nu\in\mathcal{P}(S)$,

$$\begin{aligned} d(T^*\mu, T^*\nu) &= \sup_{A\in\mathcal{A}}\{|T^*\mu(A) - T^*\nu(A)|\} \\ &\le \int_\Gamma \sup_{A\in\mathcal{A}}|\mu\gamma^{-1}(A) - \nu\gamma^{-1}(A)|P(d\gamma) \\ &\le d(\mu,\nu) \end{aligned}$$

If $A \in \mathcal{A}$, then $A = \{y \in S : \gamma_n \cdots \gamma_1(y) \leq x\}$ for some $(\gamma_1, \cdots, \gamma_n) \in \Gamma^n$ and for some $x \in R^k$. Since γ is continuous monotone, A is one of the following two types:

$$(T_1) \quad \text{if} \quad y \in A, \quad y' \in S \quad \text{such that} \quad y' \leq y \quad \text{is in} \quad A.$$

$$(T_2) \quad \text{if} \quad y \in A, \quad y' \in S \quad \text{such that} \quad y' \geq y \quad \text{is in} \quad A.$$

LEMMA 2.3. *Suppose there exists some $x_0 \in S$ and a positive integer m such that (2.2) holds. Then for any $\mu, \nu \in \mathcal{P}(S)$,*

$$d(T^{*m}\mu, T^{*m}\nu) \leq \rho d(\mu, \nu)$$

where $\rho = \max\{1 - P^m(\Gamma_1), 1 - P^m(\Gamma_2)\} < 1$.

Proof. If we let $\mathcal{R}_1 = \{y \in S : y \leq x_0\}$, $\mathcal{R}_2 = \{y \in S : y \geq x_0\}$, then $\Gamma_1 = \{\gamma \in \Gamma^m : \gamma(S) \subset \mathcal{R}_1\}$, $\Gamma_2 = \{\gamma \in \Gamma^m : \gamma(S) \subset \mathcal{R}_2\}$. Let $\mathcal{A}_1 = \{A \in \mathcal{A} : A \cap \mathcal{R}_2 \neq \emptyset\}$ and $\mathcal{A}_2 = \{A \in \mathcal{A} : A \cap \mathcal{R}_2 = \phi\}$. Divide $\mathcal{A}_1$ into three parts such as

$$\begin{aligned}
\mathcal{A}_{11} &= \{A \in \mathcal{A}_1 : A \text{ is of type}(T_1) \text{ in } (2.3)\}, \\
\mathcal{A}_{12} &= \{A \in \mathcal{A}_1 : A \text{ is of type}(T_2) \text{ and } A \cap \mathcal{R}_1 \neq \emptyset\}, \\
\mathcal{A}_{13} &= \{A \in \mathcal{A}_1 : A \text{ is of type}(T_2) \text{ and } A \cap \mathcal{R}_1 = \emptyset\}.
\end{aligned}$$

Clealy $\mathcal{A}_{11}, \mathcal{A}_{12}, \mathcal{A}_{13}, \mathcal{A}_2$ is a partition of $\mathcal{A}$.

We may easily check that if $A \in \mathcal{A}_{11}$, then $A \supset \mathcal{R}_1$ and if $A \in \mathcal{A}_{12}$, then $A \supset \mathcal{R}_2$ and hence for any $\mu, \nu \in \mathcal{P}(S)$,

$$\mu(\gamma_m \cdots \gamma_1)^{-1}(A) - \nu(\gamma_m \cdots \gamma_1)^{-1}(A) = 0$$

if

1) $A \in \mathcal{A}_{11} \cup \mathcal{A}_{13}$ and $(\gamma_1, \cdots, \gamma_m) \in \Gamma_1$ or if
2) $A \in \mathcal{A}_{12} \cup \mathcal{A}_2$ and $(\gamma_1, \cdots, \gamma_m) \in \Gamma_2$ or if
3) $(\gamma_1, \cdots, \gamma_m) \in \Gamma_1 \cap \Gamma_2$, since for each case 1), 2), 3), $(\gamma_m \cdots \gamma_1)^{-1}(A)$ becomes empty or the hole set S.

Now,

$$\begin{aligned} d(T^{*m}\mu, T^{*m}\nu) &= \sup_{A\in\mathcal{A}}\{|\int_{\Gamma^m} \mu(\gamma_m\cdots\gamma_1)^{-1}(A)dP^m(\gamma_m\cdots\gamma_1) \\ &\quad - \int_{\Gamma^m} \nu(\gamma_m\cdots\gamma_1)^{-1}(A)dP^m(\gamma_m\cdots\gamma_1)|\} \\ &\le \sup_{A\in\mathcal{A}}\{I_1+I_2+I_3+I_4\}, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{\Gamma_1-\Gamma_1\cap\Gamma_2} |\mu(\gamma_m\cdots\gamma_1)^{-1}(A) - \nu(\gamma_m\cdots\gamma_1)^{-1}(A)|dP^m(\gamma_m\cdots\gamma_1) \\ I_2 &= \int_{\Gamma_2-\Gamma_1\cap\Gamma_2} |\mu(\gamma_m\cdots\gamma_1)^{-1}(A) - \nu(\gamma_m\cdots\gamma_1)^{-1}(A)|dP^m(\gamma_m\cdots\gamma_1) \\ I_3 &= \int_{\Gamma_m-\Gamma_1\cup\Gamma_2} |\mu(\gamma_m\cdots\gamma_1)^{-1}(A) - \nu(\gamma_m\cdots\gamma_1)^{-1}(A)|dP^m(\gamma_m\cdots\gamma_1) \\ I_4 &= \int_{\Gamma_1\cap\Gamma_2} |\mu(\gamma_m\cdots\gamma_1)^{-1}(A) - \nu(\gamma_m\cdots\gamma_1)^{-1}(A)|dP^m(\gamma_m\cdots\gamma_1). \end{aligned}$$

Because I_1 vanishes on $\mathcal{A}_{11}\cup\mathcal{A}_{13}$ and I_2 vanishes on $\mathcal{A}_{12}\cup\mathcal{A}_2$,

$$\sup_{A\in\mathcal{A}}\{I_1+I_2\} \le \max\{\sup_{A\in\mathcal{A}} I_1, \sup_{A\in\mathcal{A}} I_2\}.$$

Therefore we have

$$\begin{aligned} d(T^{*m}\mu, T^{*m}\nu) &\le [\max\{P^m(\Gamma_1) - P^m(\Gamma_1\cap\Gamma_2), P^m(\Gamma_2) - P^m(\Gamma_1\cap\Gamma_2)\} \\ &\quad + 1 - P^m(\Gamma_1\cup\Gamma_2)]d(\mu,\nu) \\ &= \max\{1 - P^m(\Gamma_1), 1 - P^m(\Gamma_2)\}d(\mu,\nu). \end{aligned}$$

Our main result is the following:

THEOREM 2.4. *If (2.2) holds for some $x_0 \in S$ and some positive integer m, then there exists a unique invariant probability π on $(S, \mathcal{B}(S))$ such that*

$$\sup_{x\in S} d(P^{(n)}(x,dy), \pi(dy)) \le \rho^{[n/m]} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{2.4}$$

Proof. For $\mu, \nu \in \mathcal{P}(S)$,

$$\begin{aligned} d(T^{*n}\mu, T^{*n}\nu) &= d(T^{*m}(T^{*(n-m)}\mu), T^{*m}(T^{*(n-m)}\nu)) \\ &\leq \rho d(T^{*(n-m)}\mu, T^{*(n-m)}\nu) \\ &\leq \cdots \leq \rho^{[n/m]} d(\mu,\nu) \leq \rho^{[n/m]}, \end{aligned}$$

$n = 1, 2, 3, \cdots$, since $d(\mu,\nu) \leq 1$, $\forall \mu, \nu \in \mathcal{P}(S)$.

For $n' > n$, one has

$$\begin{aligned} d(P^{(n)}(x,dy), p^{(n')}(x,dy)) &= d(T^{*n}\mu, T^{*n}\nu) \\ &\leq \rho^{[n/m]}, \end{aligned} \tag{2.5}$$

where $\mu = \delta_x$, and $\nu = T^{*(n'-n)}\delta_x$.

Hence $P^{(n)}(x,dy)$ is a Cauchy sequence in the metric d. Let π be its limit, which exists by Lemma 2.1. Letting $n' \longrightarrow \infty$ in (2.5), we get (2.4). Since γ is continuous, $x \longrightarrow P(x,dy)$ is weak-star continuous. The fact that $x \longrightarrow P(x,dy)$ is weak-star continuous ensures that T^* on $\mathcal{P}(S)$ is weak-star continuous. The reason is: Suppose μ_n converges weakly to μ, $\mu_n, \mu \in \mathcal{P}(S)$. Then for real-valued bounded continuous function f on S,

$$\begin{aligned} \int_S f(z)(T^*\mu_n)dz &= \int_S\int_S f(z)P(x,dz)\mu_n(dx) \\ &\longrightarrow \int_S\int_S f(z)P(x,dz)\mu(dx) \\ &= \int_S f(z)(T^*\mu)(dz). \end{aligned}$$

Weak-star continuity of T^* together with weak convergence of $T^*(P^n(x, dy)$
$= P^{(n+1)}(x,dy)$ to $\pi(dy)(n \longrightarrow \infty)$ implies the invariance of π, i.e., $T^*\pi = \pi$ which completes our proof.

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