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AN EXISTENCE OF LINEAR SYSTEMS WITH GIVEN TRANSFER FUNCTION

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A vector space $\mathcal{K}$ with scalar product $< .,. >$ is called a Krein space if it can be decomposed as an orthogonal sum of a Hilbert space and an anti-space of a Hilbert space. The space $\mathcal{K}$ induces a Hilbert space $\mathcal{K}_J$ in the inner product $< .,. >_{\mathcal{K}_J} =< .,. >_{\mathcal{K}}$, where $J^2 = I$. The eigenspaces of J are denoted by $\mathcal{K}_J^+$, which is a Hilbert space and $\mathcal{K}_J^-$, which is an anti-space of a Hilbert space. Then the Krein space $\mathcal{K}$ is the orthogonal sum of $\mathcal{K}_J^+$ and $\mathcal{K}_J^-$. Such a decomposition of $\mathcal{K}$ is called a fundamental decomposition. In general, fundamental decompositions are not unique. The norm of the Hilbert space depends on the choice of a fundamental decomposion, but such norms are equivalent. The topology generated by these norms is called the strong or Mackey topology of $\mathcal{K}$. It is used to define all topological notions on the Krein space $\mathcal{K}$ with respect to this topology.

The Pontryagin index of a Krein space is the dimension of the anti-space of a Hilbert space in any such decomposition. The dimension does not depend on the choice of orthogonal decomposition. A Krein space is called a Pontryagin space if it has finite Pontryagin index.

A fixed Krein space $\mathcal{C}$ is used as a coefficient space. A vector is always an element of this space. An operator is a continuous transformation of vectors into vectors. If b is a vector, b^- is the linear functional on vectors defined by the scalar product $b^-a = \langle a, b\rangle_{\mathcal{C}}$ for every vector a. If a and b are vectors, then ab^- is the operator defined by $(ab^-)c = a(b^-c)$ for every vector c. A bar is also used for the adjoint of an operator.

A linear system is a matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ of continuous transformations on the Cartesian product Krein space $\mathcal{H} \times \mathcal{C}$ realized as a space of

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column vectors. The underlying Krein space $\mathcal{H}$ is called the state space and the auxiliary Krein space $\mathcal{C}$ is called the coefficient space or the external space.

A linear system is said to be contractive if the matrix is contractive, unitary if the matrix is unitary, and conjugate isometric if the matrix has an isometric adjoint. A linear system is said to be observable if there is no nonzero element f of the state space such that $CA^n f = 0$ for every nonnegative integer n. An observable linear system is said to be in a canonical form if the elements of the state space are power series with vector coefficients in such a way that the identity $a_n = CA^n f$ holds whenever $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

If an observable linear system is in a canonical form, then the elements of the state space are power series which converge in some neighborhood of the origin. For this linear system the main transformation A is the difference-quotient transformation, which takes $f(z)$ into $[f(z) - f(0)]/z$. The output transformation C takes $f(z)$ into $f(0)$. The input transformation B takes c into $[W(z) - W(0)]c/z$ for some power series $W(z)$ with operator coefficients which converges in a neighborhood of the origin. The external operator D is $W(0)$. The power series $W(z)$ is called the transfer function of the linear system.

The theory of canonical linear systems which are conjugate isometric is a generalization of the theory of square summable power series with vector coefficients. Assume that the coefficient space $\mathcal{C}$ is a Krein space. Write $\mathcal{C}$ as the orthogonal sum of a Hilbert space $\mathcal{C}_+$ and the anti-space $\mathcal{C}_-$ of a Hilbert space. Let J be the operator which is the identity on $\mathcal{C}_+$ and which is minus the identity on $\mathcal{C}_-$. The space $\mathcal{C}(z)$ of square summable power series is the set of power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with coefficients in $\mathcal{C}$ such that $\sum_{n=0}^{\infty} a_n^- J a_n$ is finite. The condition does not depend on the choice of decompositions of $\mathcal{C}$. The space $\mathcal{C}(z)$ is considered as a Krein space with the unique scalar product such that

$$\langle f(z), f(z) \rangle_{\mathcal{C}(z)} = \sum_{n=0}^{\infty} a_n^- a_n.$$

The construction of linear systems in Krein spaces made by Ando [1] makes use of a Krein space generalization of complementation theory [3,4].

THEOREM 1. *If a Krein space $\mathcal{P}$ is contained continuously and contractively in a Krein space $\mathcal{H}$, then a unique Krein space $\mathcal{Q}$ exists, which is contained continuously and contractively in $\mathcal{H}$, such that the inequality*

$$\langle c, c\rangle_{\mathcal{H}} \ \le\ \langle a, a\rangle_{\mathcal{P}} + \langle b, b\rangle_{\mathcal{Q}}$$

holds whenever $c = a + b$ with a in $\mathcal{P}$ and b in $\mathcal{Q}$ and such that every element c of $\mathcal{H}$ admits some such decomposition for which equality holds.

The space $\mathcal{Q}$ is called the complementary space to $\mathcal{P}$ in $\mathcal{H}$. A unique minimal decomposition is obtained when equality holds. If

$$\langle c, c\rangle_{\mathcal{H}} \ =\ \langle a, a\rangle_{\mathcal{P}} + \langle b, b\rangle_{\mathcal{Q}}$$

where $c = a + b$, then a is obtained from c under the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$ and b is obtained from c under the adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$.

Complementation theory can be used to give new proofs of theorems of Dritschel [6] and of Dritschel and Rovnyak [7] which generalize the commutant lifting theorem to Krein spaces [5].

Let $W(z)$ be a power series with operator coefficients such that multiplication by $W(z)$ is a contractive transformation in $\mathcal{C}(z)$. A construction of a canonical linear system which is contractive and conjugate isometric with $W(z)$ as its transfer function can be made using complementation theory. Multiplication by $W(z)$ in $\mathcal{C}(z)$ is continuous by the closed graph theorem. The range $\mathcal{M}(W)$ of multiplication by $W(z)$ is considered as a Krein space with the unique scalar product such that multiplication by $W(z)$ acts as a contractive partial isometry of $\mathcal{C}(z)$ onto $\mathcal{M}(W)$. The state space of the canonical linear system which is contractive and conjugate isometric and which has transfer function $W(z)$ is the complementary space $\mathcal{H}(W)$ to $\mathcal{M}(W)$ in $\mathcal{C}(z)$.

The formal adjoint of multiplication by $W(z)$ in $\mathcal{C}(z)$ is a transformation which takes polynomials into polynomials without raising degree.

We write $W(z) = \sum_{n=0}^{\infty} w_n z^n$. The formal adjoint maps $f(z)$ into g whenever $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are polynom with vector coefficients such that $b_r = \sum_{n=0}^{\infty} w_n^- a_{n+r}$ for every nonn ative integer r.

The Caratheodory-Fejer extention theory [2] can be used to constr a Krein space.

Define power series $f(z)$ and $g(z)$ to be r-equivalent for a nonnegat integer r if the power series $f(z) - g(z)$ is divisible by z^r. Let $\mathcal{C}_r$ be the Hilbert space of r-equivalence classes of power series with vec coefficients, the scalar product being defined in $\mathcal{C}_r(z)$ so that ev polynomial of degree less than r has same norm in $\mathcal{C}_r(z)$ as in $\mathcal{C}(z)$ $W(z)$ is a given power series with operator coefficients, define $\mathcal{G}_r(W$ be the graph of the adjoint of multiplication by $W(z)$ in $\mathcal{C}_r(z)$. Consi $\mathcal{G}_r(W)$ as a Hilbert space with the unique scalar product such that identity

$$\begin{aligned} &\langle (h(z), g(z)), (h(z), g(z)) \rangle_{\mathcal{G}_r(W)} \\ = \; &\langle J\, h(z), h(z) \rangle_{\mathcal{C}_r(z)} + \langle J\, g(z), g(z) \rangle_{\mathcal{C}_r(z)} \end{aligned}$$

is satisfied.

Define $core_r(W)$ to be the space of power series with vector co cients of the form $h(z) - W(z)g(z)$ such that $(h(z), g(z))$ is in $\mathcal{G}_r($ Then $core_r(W)$ admits a unique scalar product such that the ident

$$\begin{aligned} &\langle h(z) - W(z)g(z), h(z) - W(z)g(z) \rangle_{core_r(W)} \\ = \; &\langle h(z), h(z) \rangle_{\mathcal{C}_r(z)} - \langle g(z), g(z) \rangle_{\mathcal{C}_r(z)} \end{aligned}$$

is satisfied.

A construction of a Krein space can be made using the complem tation theory.

THEOREM 2. *A Krein space $\mathcal{H}_r(W)$ exists which contains $core_r($ isometrically and is contained continuously in $\mathcal{C}_r(z)$.*

Proof. Define the Mackey topology of $\mathcal{C}_r(z)$ by

$$\langle (J + T_r J T_r^*) f(z), g(z) \rangle_{\mathcal{C}_r(z)}$$

where T_r is multiplication by $W(z)$ as a transformation in the space $\mathcal{C}_r(z)$. A transformation of H in the space $\mathcal{C}_r(z)$ into itself exists which is self-adjoint with respect to this Hilbert space scalar product such that

$$\begin{aligned}&\langle (J + T_r J T_r^*) H f(z), g(z)\rangle_{\mathcal{C}_r(z)}\\ &= \langle (I - T_r T_r^*) f(z), g(z)\rangle_{\mathcal{C}_r(z)}\end{aligned}$$

is satisfied. By the spectral theorem for self-adjoint transformations, there are unique closed subspaces $\mathcal{M}_+$, $\mathcal{M}_0$ and $\mathcal{M}_-$ which are invariant under the action of H such that the restriction of H to $\mathcal{M}_+$ has nonnegative spectrum, $\mathcal{M}_0$ is the kernel of H, the restriction of H to $\mathcal{M}_-$ has nonpositive spectrum and $\mathcal{C}_r(z)$ is orthogonal sum of $\mathcal{M}_+$, $\mathcal{M}_0$ and $\mathcal{M}_-$ for the Hilbert scalar product. The space $core_r(W)$ is the orthogonal sum of a space $core_r^+(W)$, which is the image of $\mathcal{M}_+$ under $I - T_r T_r^*$, and a space $core_r^-(W)$, which is the image of $\mathcal{M}_-$ under $I - T_r T_r^*$. By the complementation theory for a self-adjoint transformation, a unique Hilbert space $\mathcal{H}_r^+(W)$ exists, which contains $core_r^+(W)$ isometrically and which is contained continuously in the space $\mathcal{C}_r(z)$ such that the adjoint of the inclusion of $\mathcal{H}_r^+(W)$ in $\mathcal{C}_r(z)$ coinsides with the adjoint of the inclusion of $core_r^+(W)$ in the space $\mathcal{C}_r(z)$. A unique Hilbert space $\mathcal{H}_r^-(W)$ exists, which contains $core_r^-(W)$ isometrically and which is contained continuously in the space $\mathcal{C}_r(z)$ such that the adjoint of the inclusion of $\mathcal{H}_r^-(W)$ in $\mathcal{C}_r(z)$ coincides with the adjoint of the inclusion of $core_r^-(W)$ in the space $\mathcal{C}_r(z)$. A unique Krein space $\mathcal{H}_r(W)$ exists, which is contained continuously in $\mathcal{C}_r(z)$ and which is the orthogonal sum of $\mathcal{H}_r^+(W)$ and $\mathcal{H}_r^-(W)$. This completes the proof of the theorem. □

Let $A(z)$ be a power series with operator coefficients such that multiplication by $A(z)$ is contractive in the Krein space $\mathcal{C}(z)$ and $A(0)$ is invertible. There exists the Krein space $\mathcal{H}(A)$ which is the state space of a canonical conjugate-isometric linear system with transfer function $A(z)$. Consider the power series $B(z) = (A(z))^{-1}$. Construct a Krein space $\mathcal{H}(B)$ which is the state space of a canonical linear system whose transfer function is $B(z)$.

Let T_r be multiplication by $B(z)$ as a transformation in the space $\mathcal{C}_r(z)$ and S_r be multiplication by $A(z)$ as a transformation in the space

$\mathcal{C}_r(z)$.

A characterization of the Krein space is the consequence of Theorem 2.

THEOREM 3. *The Krein space $\mathcal{H}_r(B)$ is the set of r-equivalence of power series with vector coefficients of the form $B(z)f(z)$ where $f(z)$ is in the space $\mathcal{H}_r(A)$. The identity*

$$\langle B(z)f(z), B(z)f(z)\rangle_{\mathcal{H}_r(B)} = -\langle f(z), f(z)\rangle_{\mathcal{H}_r(A)}$$

holds for every element $f(z)$ in $\mathcal{H}_r(A)$.

Proof. Let $f(z)$ be in $core_r(A)$. There exists $h(z)$ in the space $\mathcal{C}_r(z)$ such that $f(z) = (I - S_r S_r^*)h(z)$. The identity

$$\begin{aligned} T_r f(z) &= T_r(I - S_r S_r^*)h(z) \\ &= T_r h(z) - S_r^* h(z) \\ &= T_r T_r^* S_r^* h(z) - S_r^* h(z) \\ &= -(I - T_r T_r^*)S_r^* h(z) \end{aligned}$$

implies that $B(z)f(z)$ is an element of $\mathcal{H}_r(B)$ for every element $f(z)$ in $\mathcal{H}_r(A)$ since $core_r(A)$ is contained isometrically and densely in $\mathcal{H}_r(A)$.

The identity

$$\begin{aligned} (I - T_r T_r^*)h(z) &= T_r S_r h(z) - T_r T_r^* h(z) \\ &= T_r(S_r S_r^* T_r^* h(z) - T_r^* h(z)) \\ &= -T_r(I - S_r S_r^*)T_r^* h(z) \end{aligned}$$

implies that every element in the space $\mathcal{H}_r(B)$ can be written by $B(z)f(z)$ for some $f(z)$ in the space $\mathcal{H}_r(A)$ since $core_r(B)$ is contained isometrically and densely in $\mathcal{H}_r(B)$.

Let $f(z)$ be an element of $core_r(A)$. There exists a power series $h(z)$

in the space $\mathcal{C}_r(z)$ such that $f(z) = (I - S_r S_r^*)h(z)$. The identity

$$\begin{aligned}
\langle T_r f(z), T_r f(z)\rangle_{\mathcal{H}_r(B)} &= \langle (I - T_r T_r^*)S_r^* h(z), (I - T_r T_r^*)S_r^* h(z)\rangle_{\mathcal{H}_r(B)} \\
&= \langle (I - T_r T_r^*)S_r^* h(z), S_r^* h(z)\rangle_{\mathcal{C}_r(z)} \\
&= \langle S_r^* h(z), S_r^* h(z)\rangle_{\mathcal{C}_r(z)} - \langle T_r h(z), S_r^* h(z)\rangle_{\mathcal{C}_r(z)} \\
&= \langle S_r^* S_r^* h(z), h(z)\rangle_{\mathcal{C}_r(z)} - \langle h(z), h(z)\rangle_{\mathcal{C}_r(z)} \\
&= -\langle (I - S_r S_r^*)h(z), h(z)\rangle_{\mathcal{C}_r(z)} \\
&= -\langle f(z), f(z)\rangle_{\mathcal{H}_r(A)}
\end{aligned}$$

is satisfied. This completes the proof of the theorem. □

Let $\mathcal{H}$ be the state space of a canonical linear system which is conjugate isometric with transfer function $W(z)$. The augumented space $\mathcal{H}'$ is the set of power series $f(z)$ with vector coefficients such that $[f(z) - f(0)]/z$ belongs to $\mathcal{H}$. Equivalently the elements of $\mathcal{H}'$ are the power series of the form $c + zf(z)$ with $f(z)$ in $\mathcal{H}$ and c in $\mathcal{C}$. The space $\mathcal{H}'$ becomes a Krein space when considered with the Cartesian scalar product Krein space $\mathcal{H} \times \mathcal{C}$. This is the unique scalar procduct for which the identity for the difference-quotients

$$\begin{aligned}
&\langle f(z), f(z)\rangle_{\mathcal{H}'} \\
&= \langle [f(z) - f(0)]/z, [f(z) - f(0)]/z\rangle_{\mathcal{H}} + f(0)^- f(0)
\end{aligned}$$

holds for every element f(z) in $\mathcal{H}'$. In this notation the matrix of the canonical linear system with the state space $\mathcal{H}$ and transfer function $W(z)$ is isomorpic to the transformation of $\mathcal{H}'$ into itself which takes $f(z)$ into $[f(z) - f(0)]/z + W(z)f(0)$. The transformation has an isometric adjoint by hypothesis. An equivalent condition is that a partially isometric transformation of the Cartesian product Krein space $\mathcal{H} \times \mathcal{C}$ onto $\mathcal{H}'$ is defined by taking a pair $(f(z), c)$ into $f(z) + W(z)c$. Explicitly this means that $\mathcal{H}$ is contained continulusly in $\mathcal{H}'$ and that multiplication by $W(z)$ is a continuous transformation of $\mathcal{C}$ into $\mathcal{H}'$.

Every element of $\mathcal{H}'$ is of the form $f(z) + W(z)c$ with $f(z)$ an element of $\mathcal{H}$ and c an element of $\mathcal{C}$. The identity

$$\begin{aligned}
&\langle f(z) + W(z)c, f(z) + W(z)c\rangle_{\mathcal{H}'} \\
&= \langle f(z), f(z)\rangle_{\mathcal{H}} + c^- c
\end{aligned}$$

holds if and only if the identity $\langle f(z), W(z)k\rangle_{\mathcal{H}} = k^{-} c$ holds for every vector k such that $W(z)k$ belongs to $\mathcal{H}$.

An existence of the state space of a canonical conjugate-isometric linear system can be shown.

THEOREM 4. *There is a partial isometry from the Cartician product Krein space $\mathcal{H}_{r-1}(B) \times \mathcal{C}$ onto $\mathcal{H}_r(B')$ with $B'(z) = zB(z)$ which takes $(f(z), c)$ into $f(z) + B(z)c$.*

Proof. Let $f(z)$ be in $\mathcal{H}_{r-1}(B)$. $f(z)$ can be written as $f(z) = B(z)g(z)$ for some $g(z)$ in $\mathcal{H}_{r-1}(A)$. By complementation theory the identity

$$\begin{aligned}&\langle g(z) + A(z)a, g(z) + A(z)a\rangle_{\mathcal{H}_r(A')}\\ &= \langle g(z), g(z)\rangle_{\mathcal{H}_{r-1}(A)} + a^{-} a\end{aligned}$$

holds if, and only if, the identity

$$\langle g(z), A(z)b\rangle_{\mathcal{H}_{r-1}(A)} = b^{-} a$$

holds for every vector a such that $A(z)b$ belongs to $\mathcal{H}^{r-1}(A)$. It implies that the identity

$$\begin{aligned}&\langle f(z) + B(z)c, f(z) + B(z)c\rangle_{\mathcal{H}_r(B')}\\ =&\langle [f(z) - f(0)]/z + [B(z) - B(0)]c/z,\\ &\quad [f(z) - f(0)]/z + [B(z) - B(0)]c/z\rangle_{\mathcal{H}_{r-1}(B)}\\ &+ (f(0) + B(0)c)^{-}(f(0) + B(0)c)\\ =& - \langle g(z) - A(z)(f(0) + B(0)c), g(z) - A(z)(f(0) + B(0)c)\rangle_{\mathcal{H}_{r-1}(A)}\\ &+ (f(0) + B(0)c)^{-}(f(0) + B(0)c)\\ =& - \langle g(z), g(z)\rangle_{\mathcal{H}_{r-1}(A)} + c^{-} c\\ =&\langle f(z), f(z)\rangle_{\mathcal{H}_{r-1}(B)} + c^{-} c\end{aligned}$$

is satisfied whenever

$$\langle f(z), B(z)k\rangle_{\mathcal{H}_{r-1}(B)} = k^{-} c$$

for every $B(z)k$ in $\mathcal{H}^{r-1}(B)$. This completes the proof of the theorem. □

Let $\mathcal{H}(B)$ be the set of a power series $f(z)$ such that $f(z)$ is r-equivalent to an element in the space $\mathcal{H}_r(B)$ for every r. The space $\mathcal{H}(B)$ becomes a Krein space when considered with the unique scalar product

$$\langle f(z), f(z)\rangle_{\mathcal{H}(B)} = \lim_{r\to\infty} \langle f(z), f(z)\rangle_{\mathcal{H}_r(B)}.$$

Then the Krein space $\mathcal{H}(B)$ is the state space of a canonical linear system which is conjugate isometric with transfer function $B(z)$.

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