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ON UNIFORMLY ULTRASEPARATING FAMILY OF FUNCTION ALGEBRAS

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1. Introduction

Let X be a compact Hausdorff space, and let $C(X)$ (resp. $C_{\mathbf{R}}(X)$) be the complex (resp. real) Banach algebra of all continuous complex-valued (resp. real-valued) functions on X with the pointwise operations and the supremum norm $\| \ \|_X$. A *Banach function algebra on* X is a Banach algebra lying in $C(X)$ which separates the points of X and contains the constants. A Banach function algebra on X equipped with the supremum norm is called a *uniform algebra on* X, that is, a uniformly closed subalgebra of $C(X)$ which separates the points of X and contains the constants.

Let E be a (real or complex) normed linear space with norm $\| \ \|_E$. Denote by $\tilde{E} = \ell^\infty(\mathrm{N}, E)$ the space of all bounded functions from the set $\mathrm{N} = \{1, 2, 3, \dots\}$ to E normed as follows:

$$\|\tilde{f}\|_{\tilde{E}} \equiv \sup\{\|f_n\|_E : n \in \mathrm{N}\} < \infty$$

for a sequence $\tilde{f} = \{f_n\}_{n=1}^\infty$ in $\tilde{E}$.

Denote by $\tilde{X} = \beta(\mathrm{N} \times X)$ the Stone-Čech compactification of the product space $\mathrm{N} \times X$. Since every sequence $\{f_n\}_{n=1}^\infty$ in $\ell^\infty(\mathrm{N}, C(X))$ can be considered as a function from $\mathrm{N} \times X$ to $\mathbf{C}$, it has a unique continuous extension to a function in $C(\tilde{X})$. So, we have $\ell^\infty(\mathrm{N}, C(X)) = C(\tilde{X})$.

DEFINITION 1.1 ([2]). Let E be a (real or complex) normed linear space continuously injected in $C(X)$. We say that E is ultraseparating on X if $\tilde{E}$ separates the points of $\tilde{X}$.

Let A be a Banach function algebra on X. Denote by ball$A$ the set of all functions in A with $\|f\|_A \leq 1$.

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PROPOSITION 1.2 ([1]). *Let A be a Banach function algebra on X. Then the following are equivalent:*

(1) *A is ultraseparating on X.*

(2) *There exist a $\delta > 0$ and an $n \in \mathbb{N}$ such that for any pair of non-empty disjoint closed subsets E and F of X, there exist $u_1, \ldots, u_n; v_1, \ldots, v_n \in$ ball$A$ such that*

$$\sum_{i=1}^{n}(|u_i| - |v_i|) \geq \delta \quad \text{on} \quad E,$$
$$\sum_{i=1}^{n}(|u_i| - |v_i|) \leq -\delta \quad \text{on} \quad F.$$

Notice that if A is ultraseparating for some $\delta > 0$ and $n \in \mathbb{N}$, then A is ultraseparating for any positive integer $m \geq n$ with same $\delta > 0$. So, we may assume that $\delta \geq 1/n$, and hence we can redefine ultraseparability of a Banach function algebra as follows:

DEFINITION 1.3. A Banach function algebra A on X is said to be n-ultraseparating on X for some $n \in \mathbb{N}$ if for any pair of non-empty disjoint closed subsets E and F of X, there exist $u_1, \ldots, u_n; v_1, \ldots, v_n \in$ ball$A$ such that

$$\sum_{i=1}^{n}(|u_i| - |v_i|) \geq 1/n \quad \text{on} \quad E,$$
$$\sum_{i=1}^{n}(|u_i| - |v_i|) \leq -1/n \quad \text{on} \quad F.$$

DEFINITION 1.4. Let Γ be an index set, and let A_γ be an n_γ-ultraseparating Banach function algebras on X_γ for $\gamma \in \Gamma$. The family $\{A_\gamma : \gamma \in \Gamma\}$ is said to be uniformly ultraseparating if $\sup\{n_\gamma : \gamma \in \Gamma\} < \infty$.

Now, let A_γ be a Banach function algebra on X_γ for each $\gamma \in \Gamma$, and let $X = \bigcup_{\gamma\in\Gamma}(\{\gamma\} \times X_\gamma)$. Define an indexed family $\tilde{f} = \{f_\gamma\}_{\gamma\in\Gamma}$

of functions $f_\gamma \in A_\gamma$ by

$$\tilde{f}(\gamma, x_\gamma) = f_\gamma(x_\gamma) \quad \text{for} \quad (\gamma, x_\gamma) \in \{\gamma\} \times X_\gamma.$$

Let $\tilde{A}$ be the set of all $\tilde{f} = \{f_\gamma\}_{\gamma\in\Gamma}$ such that $\sup\{\|f_\gamma\|_{A_\gamma} : \gamma \in \Gamma\} < \infty$. Then $\tilde{A}$ is a Banach algebra (with the norm $\|\tilde{f}\|_{\tilde{A}} = \sup\{\|f_\gamma\|_{A_\gamma} : \gamma \in \Gamma\}$) lying in $C(\beta X)$ which contains the constants. Indeed, $\|\tilde{f}\|_{\tilde{A}} \geq \|\tilde{f}\|_{\beta X}$ because $\|\tilde{f}\|_{\beta X} = \sup\{\|f_\gamma\|_{X_\gamma} : \gamma \in \Gamma\}$.

In this paper, we will study point separability and ultraseparability of the algebra $\tilde{A}$ *via* uniform ultraseparability of the family $\{A_\gamma : \gamma \in \Gamma\}$.

2. Main Results

LEMMA 2.1 ([3]). *Let A be a real Banach function algebra on X. Then the space of all linear combinations of $|f|$ for $f \in ballA$ is uniformly dense in $C_{\mathbf{R}}(X)$.*

THEOREM 2.2. *Let A_γ be a Banach function algebra on X_γ for each $\gamma \in \Gamma$. Then the following are equivalent:*

(1) *$\tilde{A}$ separates the points of βX.*
(2) *There exists a finite subset Γ_0 of Γ such that the family $\{A_\gamma : \gamma \in \Gamma - \Gamma_0\}$ is uniformly ultraseparating.*

Proof. (2) $\Longrightarrow$ (1): Let Γ_o be a finite subset of Γ such that the family $\{A_\gamma : \gamma \in \Gamma - \Gamma_o\}$ is uniformly ultraseparating. Let p and q be distinct two points of βX. Put

$$Y_1 = \bigcup_{\gamma\in\Gamma_0} (\{\gamma\} \times X_\gamma) \quad \text{and} \quad Y_2 = X - Y_1.$$

Then $\beta X = \beta(Y_1 \cup Y_2) = \beta Y_1 \cup \beta Y_2 = Y_1 \cup \beta Y_2$ since Γ_0 is a finite set.

Case 1. p and $q \in Y_1$:
If $p, q \in \{\gamma_0\} \times X_{\gamma_0}$ for some $\gamma_0 \in \Gamma_0$, then $p = (\gamma_0, x)$, $q = (\gamma_0, y)$ for some distinct $x, y \in X_{\gamma_0}$. Since A_{γ_0} separates the points of X_{γ_0}, there

exists $g \in A_{\gamma_0}$ such that $g(x) \neq g(y)$. Choose any $\tilde{f} = \{f_\gamma\}_{\gamma\in\Gamma} \in \tilde{A}$ such that $f_{\gamma_0} = g$. Then $\tilde{f}|_{\{\gamma_0\}\times X_{\gamma_0}} = f_{\gamma_0}$, and

$$\tilde{f}(p) = f_{\gamma_0}(x) \neq f_{\gamma_0}(y) = \tilde{f}(q).$$

Next, suppose that $p = (\gamma_0, x_{\gamma_0}) \in \{\gamma_0\} \times X_{\gamma_0}$, $q = (\gamma_1, x_{\gamma_1}) \in \{\gamma_1\} \times X_{\gamma_1}$ for some distinct $\gamma_0, \gamma_1 \in \Gamma_0$. Define for $\gamma \in \Gamma$,

$$f_\gamma = \begin{cases} 0 & \text{if} \quad \gamma = \gamma_0, \\ 1 & \text{if} \quad \gamma \neq \gamma_0. \end{cases}$$

Then $\tilde{f} = \{f_\gamma\}_{\gamma\in\Gamma} \in \tilde{A}$, and

$$\tilde{f}(p) = f_{\gamma_0}(x_{\gamma_0}) = 0 \neq 1 = f_{\gamma_1}(x_{\gamma_1}) = \tilde{f}(q).$$

Case 2. $p \in Y_1, q \in \beta Y_2$ (or $p \in \beta Y_2, q \in Y_1$):
Assume that $p \in Y_1, q \in \beta Y_2$. Then $p = (\gamma_0, x_0)$ for some $\gamma_0 \in \Gamma_0$ and $x_0 \in X_{\gamma_0}$. Define

$$f_\gamma = \begin{cases} 0 & \text{if} \quad \gamma \in \Gamma - \Gamma_0, \\ 1 & \text{if} \quad \gamma \in \Gamma_0, \end{cases}$$

and let $\tilde{f} = \{f_\gamma\}_{\gamma\in\Gamma}$. Then $\tilde{f} \in \tilde{A}$, $\tilde{f}|_{\beta Y_2} = 0$, and

$$\tilde{f}(p) = f_{\gamma_0}(x_0) = 1 \neq 0 = \tilde{f}(q).$$

Case 3. $p, q \in \beta Y_2$:
It suffices to show that $\tilde{A}|_{\beta Y_2}$ separates the points of βY_2. Indeed, if then, we can choose $\tilde{g} = \{g_\gamma\}_{\gamma\in\Gamma-\Gamma_0} \in \tilde{A}|_{\beta Y_2}$ such that $\tilde{g}(p) \neq \tilde{g}(q)$. Define

$$f_\gamma = \begin{cases} g_\gamma & \text{if} \quad \gamma \in \Gamma - \Gamma_0, \\ 0 & \text{if} \quad \gamma \in \Gamma_0, \end{cases}$$

and let $\tilde{f} = \{f_\gamma\}_{\gamma\in\Gamma}$. Then $\tilde{f} \in \tilde{A}$ and $\tilde{f}|_{\beta Y_2} = \tilde{g}$. Hence, we have

$$\tilde{f}(p) = \tilde{g}(p) \neq \tilde{g}(q) = \tilde{f}(q).$$

Now, to prove that $\tilde{A}|_{\beta Y_2}$ separates the points of βY_2, let p and q be distinct two points of βY_2, and let U and V be neighborhoods of p and q, respectively in βY_2 having disjoint closures. Put

$$U^\gamma = \overline{(\{\gamma\} \times X_\gamma) \cap U}, \quad V^\gamma = \overline{(\{\gamma\} \times X_\gamma) \cap V}$$

for $\gamma \in \Gamma - \Gamma_0$. (Here, $\overline{S}$ is the closure of S in βY_2.) Then U^γ and V^γ are disjoint closed sets in $\{\gamma\} \times X_\gamma$ which is homeomorphic to X_γ for each $\gamma \in \Gamma - \Gamma_0$. Since $\{A_\gamma : \gamma \in \Gamma - \Gamma_0\}$ is uniformly ultraseparating, we can choose $n \in \mathbf{N}$ such that for each $\gamma \in \Gamma - \Gamma_0$, there exist $u_1^\gamma, \ldots, u_n^\gamma; v_1^\gamma, \ldots, v_n^\gamma \in \mathrm{ball} A_\gamma$ such that

$$\sum_{i=1}^{n}(|u_i^\gamma| - |v_i^\gamma|) \geq 1/n \quad \text{on} \quad U^\gamma,$$
$$\sum_{i=1}^{n}(|u_i^\gamma| - |v_i^\gamma|) \leq -1/n \quad \text{on} \quad V^\gamma.$$

Put $\tilde{u_i} = \{u_i^\gamma\}_{\gamma \in \Gamma - \Gamma_0}, \tilde{v_i} = \{v_i^\gamma\}_{\gamma \in \Gamma - \Gamma_0}$ for $i = 1, \ldots, n$. Then $\tilde{u_i}, \tilde{v_i} \in \mathrm{ball}(\tilde{A}|_{\beta Y_2})$ for $i = 1, \ldots, n$, and

$$(*) \qquad \begin{cases} \displaystyle\sum_{i=1}^{n}(|\tilde{u}_i| - |\tilde{v}_i|) \geq 1/n \quad \text{on} \quad \overline{U}, \\ \displaystyle\sum_{i=1}^{n}(|\tilde{u}_i| - |\tilde{v}_i|) \leq -1/n \quad \text{on} \quad \overline{V} \end{cases}$$

because $X \cap U$ and $X \cap V$ are dense in $\overline{U}$ and $\overline{V}$, respectively.

Now, if $\tilde{f}(p) = \tilde{f}(q)$ for all $\tilde{f} \in \tilde{A}|_{\beta Y_2}$, then $(*)$ is impossible. Therefore, $\tilde{A}|_{\beta Y_2}$ separates the points of βY_2.

(1) $\Longrightarrow$ (2): Suppose that $\tilde{A}$ separates the points of βX, and suppose that there is no finite subset of Γ_0 of Γ such that $\{A_\gamma : \gamma \in \Gamma - \Gamma_0\}$ is uniformly ultraseparating. For each $n \in \mathbf{N}$, put

$$\Gamma_n = \{\gamma \in \Gamma : A_\gamma \text{ is not } n\text{-ultraseparating}\}.$$

Then Γ_n must be an infinite set for each $n \in \mathrm{N}$, and $\Gamma_1 \supset \Gamma_2 \supset \dots$. Take $\gamma_1, \gamma_2, \dots \in \Gamma$ distinct so that $\gamma_n \in \Gamma_n$. Then for each $n \in \mathrm{N}$, we can choose disjoint closed sets E_n and F_n in X_{γ_n} such that for any $f_1, \dots, f_n; g_1, \dots, g_n \in \mathrm{ball} A_{\gamma_n}$, either

$$\sum_{i=1}^{n}(|f_i| - |g_i|) < 1/n \quad \text{on} \quad E_n$$

or

$$\sum_{i=1}^{n}(|f_i| - |g_i|) > -1/n \quad \text{on} \quad F_n.$$

Let $E = \overline{\bigcup_{n=1}^{\infty}(\{\gamma_n\} \times E_n)}$ and $F = \overline{\bigcup_{n=1}^{\infty}(\{\gamma_n\} \times F_n)}$. (Here, $\overline{S}$ is the closure of S in βX.) Then E and F are disjoint closed sets in βX. Let $\tilde{h}$ be a real-valued continuous function from βX onto [-1,1] such that $\tilde{h}|_E = 1, \tilde{h}|_F = -1$. Since $\tilde{A}$ separates the point of βX, so does Re$\tilde{A}$. Hence by the above Lemma 2.1, there are a positive integer N and $\tilde{u_i} = \{u_i^\gamma\}_{\gamma \in \Gamma}, \tilde{v_i} = \{v_i^\gamma\}_{\gamma \in \Gamma} \in \tilde{A}$ such that

$$\left|\sum_{i=1}^{N}(|\tilde{u_i}| - |\tilde{v_i}|) - \tilde{h}\right| \leq 1/2 \quad \text{on} \quad \beta X.$$

So, for any $n \in \mathrm{N}$, we have

$$\left|\sum_{i=1}^{N}(|u_i^{\gamma_n}| - |v_i^{\gamma_n}|) - 1\right| \leq 1/2 \quad \text{on} \quad E_n$$

and

$$\left|\sum_{i=1}^{N}(|u_i^{\gamma_n}| - |v_i^{\gamma_n}|) + 1\right| \leq 1/2 \quad \text{on} \quad F_n.$$

Take $c = \max_{1\le i\le N}\{\|\tilde{u_i}\|, \|\tilde{v_i}\|\}$. Then for $n \in \mathbf{N}$ with $1/2c \geq 1/n$, we have

$$\sum_{i=1}^{N}(|f_i^{\gamma_n}| - |g_i^{\gamma_n}|) \geq 1/n \quad \text{on} \quad E_n$$

and

$$\sum_{i=1}^{N}(|f_i^{\gamma_n}| - |g_i^{\gamma_n}|) \leq -1/n \quad \text{on} \quad F_n,$$

where $f_i^\gamma = (1/c)u_i^\gamma$ and $g_i^\gamma = (1/c)v_i^\gamma$ for $\gamma \in \Gamma$. But this is impossible. □

THEOREM 2.3. *Let A_γ be a Banach function algebra on X_γ for each $\gamma \in \Gamma$. Then the following are equivalent:*

(1) *The family $\{A_\gamma : \gamma \in \Gamma\}$ is uniformly ultraseparating.*
(2) *$\tilde{A}$ is an ultraseparating Banach function algebra on βX.*

Proof. (1) $\Longrightarrow$ (2): Suppose that $\{A_\gamma : \gamma \in \Gamma\}$ is uniformly ultraseparating. Then there exists $n \in \mathbf{N}$ such that each A_γ is n-ultraseparating on X_γ. Let E and F be disjoint closed subsets of βX with non-empty interior, and put

$$E^\gamma = (\{\gamma\} \times X_\gamma) \cap E, \quad F^\gamma = (\{\gamma\} \times X_\gamma) \cap F$$

for each $\gamma \in \Gamma$. Then E^γ and F^γ are non-empty disjoint closed sets in $\{\gamma\} \times X_\gamma$ for each $\gamma \in \Gamma$. So, for each $\gamma \in \Gamma$, we can choose $u_1^\gamma, \ldots, u_n^\gamma; v_1^\gamma, \ldots, v_n^\gamma \in \text{ball}A_\gamma$ such that

$$\sum_{i=1}^{n}(|u_i^\gamma| - |v_i^\gamma|) \geq 1/n \quad \text{on} \quad E^\gamma,$$
$$\sum_{i=1}^{n}(|u_i^\gamma| - |v_i^\gamma|) \leq -1/n \quad \text{on} \quad F^\gamma.$$

Take $\tilde{u_i} = \{u_i^\gamma\}_{\gamma\in\Gamma}, \tilde{v_i} = \{v_i^\gamma\}_{\gamma\in\Gamma}$ for $i = 1,\ldots,n$. Then $\tilde{u_i}, \tilde{v_i} \in \mathrm{ball}\tilde{A}$ for $i = 1,\ldots,n$, and

$$\sum_{i=1}^{n}(|\tilde{u_i}| - |\tilde{v_i}|) \geq 1/n \quad \text{on} \quad E,$$

$$\sum_{i=1}^{n}(|\tilde{u_i}| - |\tilde{v_i}|) \leq -1/n \quad \text{on} \quad F.$$

Therefore, $\tilde{A}$ is n-ultraseparating on βX.

(2) $\Longrightarrow$ (1): Suppose $\tilde{A}$ is n-ultraseparating on βX, and fix $\gamma_0 \in \Gamma_0$. We will show that $A_{\gamma_0} = \tilde{A}|_{\{\gamma_0\}\times X_{\gamma_0}}$ is n-ultraseparating on X_{γ_0}.

Let E and F be disjoint closed subsets of X_{γ_0}. Then $\{\gamma_0\} \times E$ and $\{\gamma_0\}\times F$ are disjoint closed sets in βX. So,there exist $\tilde{u_1},\ldots,\tilde{u_n};\tilde{v_1},\ldots,\tilde{v_n} \in \mathrm{ball}\tilde{A}$ such that

$$\sum_{i=1}^{n}(|\tilde{u_i}| - |\tilde{v_i}|) \geq 1/n \quad \text{on} \quad \{\gamma_0\} \times E,$$

$$\sum_{i=1}^{n}(|\tilde{u_i}| - |\tilde{v_i}|) \leq -1/n \quad \text{on} \quad \{\gamma_0\} \times F.$$

Denote $\tilde{u_i} = \{u_i^\gamma\}_{\gamma\in\Gamma}$, $\tilde{v_i} = \{v_i^\gamma\}_{\gamma\in\Gamma}$ for $u_i^\gamma, v_i^\gamma \in A_\gamma$, $\gamma \in \Gamma$, $i = 1,\ldots,n$. Then $u_i^{\gamma_0}$ and $v_i^{\gamma_0} \in \mathrm{ball}A_{\gamma_0}$ for $i = 1,\ldots,n$, and

$$\sum_{i=1}^{n}(|u_i^{\gamma_0}| - |v_i^{\gamma_0}|) \geq 1/n \quad \text{on} \quad E,$$

$$\sum_{i=1}^{n}(|u_i^{\gamma_0}| - |v_i^{\gamma_0}|) \leq -1/n \quad \text{on} \quad F.$$

Thus, A_{γ_0} is n-ultraseparating on X_{γ_0}, and therefore $\{A_\gamma : \gamma \in \Gamma\}$ is uniformly ultraseparating. $\square$

Let B be a Banach function algebra on a compact Hausdorff space Y. In the notations of the above theorem, take $\Gamma = \mathrm{N}$, $A_\gamma = B$, and $X_\gamma = Y$ for every $\gamma \in \Gamma$. Then $X = \bigcup_{n\in\mathrm{N}}(\{n\} \times X_n) = \mathrm{N} \times Y$, so $\beta X = \beta(\mathrm{N} \times Y) = \tilde{Y}$ and $\tilde{A} = \ell^\infty(\mathrm{N}, B) = \tilde{B}$. Thus, we have

COROLLARY 2.4 ([1]). *Let B be a Banach function algebra on a compact Hausdorff space Y. Then the following are equivalent:*

(1) *B is ultraseparating on Y.*
(2) *$\tilde{B}$ is ultraseparating on $\tilde{Y}$.*

EXAMPLE 2.5. Let $X = [0, 1]$. For each $n \in \mathrm{N}$, define

$$\|f\|_n = \|f\|_X + n|f(0)| \quad \text{for} \quad f \in C(X),$$

where $\|\ \|_X$ is the supremum norm. Then $C(X)$ equipped with the norm $\|\ \|_n$ is a Banach function algebra on X, which we denote by A_n for $n \in \mathrm{N}$. Since $\|f\|_X \leq \|f\|_n \leq (n+1)\|f\|_X$ for $f \in C(X)$ and since the uniform algebra $C(X)$ is 1-ultraseparating, A_n is $(n+1)$-ultraseparating for each $n \in \mathrm{N}$.

Let $\tilde{A}$ be the Banach algebra so defined as in Definition 1.4, and let $C(\tilde{X}) = \ell^\infty(\mathrm{N}, C(X))$ with the uniform norm on $\tilde{X} = \beta(\mathrm{N} \times X)$. Then, we have

$$\tilde{f} = \{f_n\}_{n=1}^\infty \in \tilde{A} \quad \text{iff} \quad \tilde{f} \in C(\tilde{X}) \quad \text{and} \quad \sup_{n \in \mathrm{N}} n|f_n(0)| < \infty.$$

Hence $\tilde{A}$ is a proper subalgebra of $C(\tilde{X})$, and therefore by Theorem 2.3 the family $\{A_n : n \in \mathrm{N}\}$ is not uniformly ultraseparating. Actually, we can show this fact directly as follows:

Let k be any positive integer less than $\sqrt{n+1}$. Then $|f(0)| \leq 1/(n+1)$ for $f \in \mathrm{ball}A_n$. Take $E = \{0\}$, $F = \{1\}$. If $f_1, \ldots, f_k; g_1, \ldots, g_k \in \mathrm{ball}A_n$, then we have

$$\sum_{i=1}^{k}(|f_i(0)| - |g_i(0)|) \leq \sum_{i=1}^{k} |f_i(0)| \leq \frac{k}{n+1} < \frac{\sqrt{n+1}}{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{k}.$$

Thus, A_n is not k-ultraseparating, and therefore A_n is k_n-ultraseparating for a positive integer $k_n \geq \sqrt{n+1}$ for each $n \in \mathrm{N}$. This implies that the family $\{A_n : n \in \mathrm{N}\}$ is not uniformly ultraseparating because $k_n \to \infty$ as $n \to \infty$.

REMARK. Let $A_\gamma = C(X_\gamma)$ with the uniform norm on X_γ for each $\gamma \in \Gamma$. Then $\tilde{A}$ is an ultraseparating uniform algebra on βX by Theorem 2.3 and by the fact that $\|\tilde{f}\|_{\tilde{A}} = \|\tilde{f}\|_{\beta X}$ for $\tilde{f} \in \tilde{A}$. Since each A_γ is self-adjoint, so is $\tilde{A}$, and therefore $\tilde{A} = C(\beta X)$ by the Stone-Weierstrass Theorem.

But this is not true if $A_\gamma = C(X_\gamma)$ is equipped with any other norm than the supremum norm as in the above example.

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