

(E)-SUPER-DECOMPOSABLE OPERATORS

JAE-CHUL RHO AND JONG-KWANG YOO

1. Introduction

The decomposable operator theory was introduced by C. Foias in the 1960's as an extension of the spectral operator theory which was developed for a period of over thirty years since 1940's, and since then it has been studied as one of the key research topics. In 1979, E. Albrecht discovered that the definition of decomposable operator can be replaced by a more weakened condition. Weaker conditions then appeared in the decomposable operator are weakly decomposable operator and analytically decomposable operator, and stronger conditions then appeared in the decomposable operator are strongly decomposable operator. Many research results were produced on these theories. At the same time, theories on classification of invariant subspace problem were developed.

In this paper, motivated by [7] we introduces the notion of an (E)-super-decomposable operator which is a type of a super-decomposable operator and solves the problems on dual operator and restriction and quotient operators which has not been solved under strongly decomposable operator theory. At the same time, some interesting research results are disclosed.

Throughout this paper we shall use the standard notions and some basic results of the theory of decomposable operators as presented in [8] and [10]. Let $\mathcal{L}(X)$ be the space of all continuous linear operators on a complex Banach space X . For an operator $T \in \mathcal{L}(X)$, $Lat(T)$ stands for the collection of all closed T -invariant linear subspaces of X . Also, $\mathcal{F}(\mathbb{C})$ stands for the family of closed subsets of \mathbb{C} . T^* denotes the dual operator of $T \in \mathcal{L}(X)$. If Y is a closed T -invariant subspace, we write $T|_Y$ for the restriction and T for the operator induced by T on X/Y . For $Y \subset X$, let Y^\perp be its annihilator in X . We use $\sigma(T)$ for spectrum of T and $\rho(T)$ for its resolvent set. We put \bar{Y} for the closure of Y in

Received May 21, 1992.

appropriate topology. An operator T has the single-valued extension property (abbr.,SVEP) if for each X -valued analytic function f defined on a region $V_f \subset \mathbb{C}$ such that $(\lambda I - T)f(\lambda) = 0$ for $\lambda \in V_f$, we have $f = 0$ on V_f .

If $T \in \mathcal{L}(X)$ has the single-valued extension property, we define the *local resolvent set at x* , denoted by $\rho_T(x)$, as the set of all $\lambda \in \mathbb{C}$ for which there exist an open neighborhood U of λ in \mathbb{C} and an analytic function $f : U \rightarrow X$ with $(T - \mu I)f(\mu) = x$ for all $\mu \in U$. The set $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$ will be called the *local spectrum* of T at the point $x \in X$.

If T has the single-valued extension property, then for any $F \subseteq \mathbb{C}$, the set $X_T(F) := \{x \in X \mid \sigma_T(x) \subseteq F\}$ is a linear subspace of X , hyperinvariant for T (i.e. invariant for all operators which commutes with T). Clearly, $X_T(F) = X_T(\sigma(T) \cap F)$, and $F_1 \subseteq F_2 \subseteq \mathbb{C}$ implies $X_T(F_1) \subseteq X_T(F_2)$ and in general, $X_T(F)$ is not necessarily closed even if F is closed [8].

An operator $T \in \mathcal{L}(X)$ is called *decomposable* if for every open cover $\{U, V\}$ of \mathbb{C} (or, $\sigma(T)$) there exists $Y_1, Y_2 \in \text{Lat}(T)$ such that $X = Y_1 + Y_2$ and $\sigma(T|Y_i) \subseteq U_i$ for $i = 1, 2$. It is known that decomposable operators enjoy a completely symmetric duality theory, i.e., an operator is decomposable exactly when its dual is. Also, it is known that if T is decomposable and for any closed $F \subseteq \mathbb{C}$, then $X_T(F)$ is norm closed linear subspace of X and $\sigma(T|X_T(F)) \subseteq F$. For other special properties of decomposable operators, see [8].

A linear subspace Y of X is said to be a *ν -space* for $T \in \mathcal{L}(X)$ if Y is invariant under T and $\sigma(T|Y) \subseteq \sigma(T)$. A T -invariant subspace Z is said to be *spectral maximal* for T if for any T -invariant subspace Y such that $\sigma(T|Y) \subseteq \sigma(T|Z)$, we have $Y \subseteq Z$. We denote the set of all spectral maximal spaces for T by $SM(T)$. It is known that if T is decomposable operator, then $SM(T) = \{X_T(F) : F \in \mathcal{F}(\mathbb{C})\}$.

The T -invariant subspace Y is called *analytically invariant* if for each X -valued analytic function f defined on a region V_f such that $(\lambda - T)f(\lambda) \in Y$ for $\lambda \in V_f$, then we have $f(\lambda) \in Y$ for $\lambda \in V_f$. We denote the set of all analytically invariant subspaces for T by $AI(T)$. It is known [10] that spectral maximal implies analytically invariant, but not converse.

2. Properties of (E)-super-decomposable operators

Let $J = [a, b]$ be a compact interval of the real line. Let $BV(J)$ be the Banach algebra of complex valued functions of bounded variation on J with norm $||| \cdot |||$ defined by $|||f||| = |f(b)| + \text{var}(f, J)$ ($f \in BV(J)$), where $\text{var}(f, J)$ is the total variation of f over J . Let $AC(J)$ be the Banach subalgebra of $BV(J)$ consisting of absolutely continuous functions on J . For $f \in AC(J)$,

$$|||f||| = |f(b)| + \int_a^b |f'(t)| dt.$$

Let $\mathcal{P}(J)$ be the subalgebra of $AC(J)$ consisting of the polynomials on J . Then $\mathcal{P}(J)$ is norm dense in $AC(J)$. Let $T \in \mathcal{L}(X)$. We define $p(T)$ in the natural way by setting $p(T) = \sum_{n=0}^k a_n T^n$, where $p(\lambda) = \sum_{n=0}^k a_n \lambda^n$. The $p \rightarrow p(T)$ is an algebra homomorphism. We say that T is *well-bounded operator* if there is a compact interval J and a real constant K such that $\|p(T)\| \leq K \|p\|$ for all polynomial p with complex coefficients, that is, there is compact interval J such that T has an $AC(J)$ -functional calculus. In this case we say that T is implemented by (K, J) . Observe that if T is well-bounded then so is T^* with the same J and K . It is well known [9] that if $T \in \mathcal{L}(X)$ is a well-bounded operator on X implemented by (K, J) , then $\sigma(T) \subseteq J$.

THEOREM 2.1 [7]. *Let T be a well-bounded operator on a reflexive Banach space X . Let K and J be chosen so that $\|p(T)\| \leq K \|p\|$ for all polynomial p . Then for any open cover $\{U, V\}$ of J there exists $R \in \mathcal{L}(X)$ such that $R^2 = R$, $RT = TR$, $\sigma(T|R(X)) \subseteq U$ and $\sigma(T|(I - R)(X)) \subseteq V$.*

Motivating the well-bounded operator on a reflexive Banach space which is described in Theorem 2.1, we introduce a new operator as follows:

DEFINITION 2.2. *An operator $T \in \mathcal{L}(X)$ is said to be a (E)-super-decomposable if for every pair of open sets $U, V \subseteq \mathbb{C}$ such that $U \cup V = \mathbb{C}$ (or, $\sigma(T) \subseteq U \cup V$), there exists some $R \in \mathcal{L}(X)$ such that $R^2 = R$, $RT = TR$, $\sigma(T|R(X)) \subseteq U$ and $\sigma(T|(I - R)(X)) \subseteq V$.*

The (E)-super-decomposability is replaced by the following;

An operator $T \in \mathcal{L}(X)$ is (E)-super-decomposable if for every open cover $\{U, V\}$ of \mathbb{C} , there exists idempotent operator $R \in \mathcal{L}(X)$ such that $RT = TR, \sigma(T|RX) \subseteq U$ and $\sigma(T^{RX}) \subseteq V$, where T^{RX} is a coinduced operator on the quotient space X/RX .

The following theorem gives us an equivalent condition for the (E)-super-decomposability of T which is easier to handle.

THEOREM 2.3. *The following statements are equivalent:*

- (1) $T \in \mathcal{L}(X)$ is (E)-super-decomposable
- (2) For every open covering $\{U_1, U_2\}$ of \mathbb{C} , there exists spaces $X_1, X_2 \in \text{Lat}(T)$ as well as operators $R_1, R_2 \in \mathcal{L}(X)$ commuting with T such that $R_1 + R_2 = I, R_j^2 = R_j, R_j(X) \subseteq X_j$ and $\sigma(T|X_j) \subseteq U_j$ for $j = 1, 2$.
- (3) T is decomposable and for every pair of closed invariant spaces Y and Z satisfying $\sigma(T|Y) \cap \sigma(T|Z) = \phi$, there exists some $R \in \mathcal{L}(X)$ commuting with T such that $R^2 = R, R|Y = 0$ and $(I - R)|Z = 0$.

Proof. (1) \implies (2). For every open covering $\{U_1, U_2\}$ of \mathbb{C} , there exists some $R \in \mathcal{L}(X)$ such that $R^2 = R, RT = TR, \sigma(T|R(X)) \subseteq U_1$ and $\sigma(T|(I - R)(X)) \subseteq U_2$. If we put $R_1 := R, R_2 := I - R, X_1 := R(X)$ and $X_2 := (I - R)(X)$, then one has the result.

(1) \implies (3). Clearly, T is decomposable. Let $Y, Z \in \text{Lat}(T)$ such that $\sigma(T|Y) \cap \sigma(T|Z) = \phi$. Then $U_1 := \mathbb{C} \setminus \sigma(T|Y)$ and $U_2 := \mathbb{C} \setminus \sigma(T|Z)$ are open with $U_1 \cup U_2 = \mathbb{C}$. By the assumption, there exists $X_1, X_2 \in \text{Lat}(T)$ as well as $R_1, R_2 \in \mathcal{L}(X)$ such that $R_j^2 = R_j, R_j T = TR_j, R_j(X) \subseteq X_j$ and $\sigma(T|X_j) \subseteq U_j$ for $j = 1, 2$ and $R_1 + R_2 = I$. If we put $R := R_1$, then $R^2 = R$ and $RT = TR$. We want to show that $R|Y = 0$. If $y \in Y$, then $\sigma_T(Ry) \subseteq \sigma_T(y) \subseteq \sigma(T|Y) \subseteq \mathbb{C} \setminus U_1$ by ([8], Proposition 1.2). On the other hand, we conclude from $R(X) \subseteq X_1 \subseteq X_T(\sigma(T|X_1))$ that $\sigma_T(Ry) \subseteq \sigma(T|X_1) \subseteq U_1$. It follows that $\sigma_T(Ry) = \phi$, which implies $Ry = 0$. Thus $R|Y = 0$, and the same reasoning shows that $(I - R)|Z = 0$.

(3) \implies (1). Given an arbitrary open covering U, V of \mathbb{C} , we choose open sets $U_1, U_2, V_1, V_2 \subseteq \mathbb{C}$ such that $U_1 \cup U_2 = \mathbb{C}, U_1 \subseteq \overline{U_1} \subseteq U_2 \subseteq \overline{U_2} \subseteq U$ and $V_1 \subseteq \overline{V_1} \subseteq V_2 \subseteq \overline{V_2} \subseteq V$. Then $F_1 := \mathbb{C} \setminus U_1$ and $F_2 := \mathbb{C} \setminus V_1$ are closed and disjoint. Since T is decomposable, it follows that both $X_T(F_1)$ and $X_T(F_2)$ are spectral maximal spaces with

$\sigma(T|X_T(F_1)) \cap \sigma(T|X_T(F_2)) = \phi$. Hence condition (c) supplies us with some $R \in \mathcal{L}(X)$ commuting with T such that $R^2 = R, R|X_T(F_1) = 0$ and $(I - R)|X_T(F_2) = 0$. We want to show that $\sigma(T|R(X)) \subseteq \overline{U_2}$. Since $U_2 \cup \mathbb{C} \setminus \overline{U_1} = \mathbb{C}$, we have splitting

$$X = X_T(\overline{U_2}) + X_T(\overline{\mathbb{C} \setminus \overline{U_1}}) = X_T(\overline{U_2}) + X_T(F_1).$$

Since $R|X_T(F_1) = 0$, we have $R(X) = R(X_T(\overline{U_2})) \subseteq X_T(\overline{U_2})$. Let $\lambda \in \mathbb{C} \setminus \overline{U_2}$. Consider the operator $S := [(T - \lambda)|X_T(\overline{U_2})]^{-1}$ on $X_T(\overline{U_2})$, that is, $S(T - \lambda I)|X_T(\overline{U_2}) = (T - \lambda I)S = I|X_T(\overline{U_2})$. It suffices to show that $SR(X) \subseteq R(X)$, since this implies $SR(X) \subseteq \overline{R(X)} = R(X)$, so that the restriction $S|R(X)$ will be the inverse of $(T - \lambda)|R(X)$. Given an arbitrary $x \in R(X)$, we have $x = Ry$ for some $y \in X_T(\overline{U_2})$ and hence $Sx = SRy = SR(T - \lambda)Sy = S(T - \lambda) = RSy \in R(X)$. It follows that $\sigma(T|R(X)) \subseteq \overline{U_2} \subseteq U$. By the same reasoning, $\sigma(T|(I - R)(X)) \subseteq V$.

PROPOSITION 2.4. *The approximate point spectrum of (E)-super-decomposable operators coincides with its spectrum.*

Proof. Suppose that $\sigma_{ap}(T) \neq \sigma(T)$. Then $U = \mathbb{C} \setminus \sigma(T)$ is open and $U \cap \sigma(T) \neq \phi$. Let V be a second open set such that $\{U, V\}$ cover $\sigma(T)$ and $\sigma(T) \not\subseteq V$. Then there exists $R \in \mathcal{L}(X)$ such that $R^2 = R, RT = TR, \sigma(T|R(X)) \subseteq U$ and $\sigma(T|(I - R)(X)) \subseteq V$. Thus $R(X) \neq \{0\}$, because otherwise $(I - R)(X) = X$ and $\sigma(T) = \sigma(T|(I - R)(X)) \subseteq V$ which is impossible by the choice of V . Hence there is $\lambda \in U$ such that $\lambda \in \partial\sigma(T) \subseteq \sigma_{ap}(T|R(X)) \subseteq \sigma(T)$. But this is a contradiction.

THEOREM 2.5. *If $T \in \mathcal{L}(X)$ is (E)-super-decomposable, then T^* is (E)-super-decomposable.*

Proof. For every pair of open sets U_1, U_2 of \mathbb{C} such that $U_1 \cup U_2 = \mathbb{C}$, there exists $X_1, X_2 \in \text{Lat}(T)$ as well as operators $R_1, R_2 \in \mathcal{L}(X)$ commuting with T such that $R_1 + R_2 = I, R_j^2 = R_j, R_j(X) \subseteq X_j$ and $\sigma(T|X_j) \subseteq U_j$ ($j=1,2$). Let $Y_j := X_j^*$. By Hahn Banach theorem, there exists an extension $\hat{f} \in X^*$ such that $\hat{f}|X_j = f$ and $\|\hat{f}\| = \|f\|$ for every $f \in X_j^*$. So we can identify

$$Y_j = \{\hat{f} \in X^* \mid \hat{f}|X_j = f \text{ and } f \in Y_j\}$$

for $j = 1, 2$. First, we want to show that $Y_j \in \text{Lat}(T)$. Clearly, $T^* \hat{f} = \hat{f} \circ T$ is a bounded linear functional and $Tx \in X_j$ for all $x \in X_j$. Since $(T^* \hat{f})(x) = \hat{f}(Tx) = f(Tx)$ for all $x \in X_j$, it follows that $T^* \hat{f} \in Y_j$. Hence $Y_j \in \text{Lat}(T^*)$. If $f \in X^*$, then $R_j^* f = f \circ R_j$ is a bounded linear functional. From $R_j(X) \subseteq X_j$ and $R_j^* f \in Y_j$, we have $R_j^* Y_j \subseteq Y_j$ for $j=1, 2$. It is easily check that $R_1^* + R_2^* = I^*$, $(R_j^*)^2 = R_j^*$ and $R_j^* T^* = T^* R_j^*$ for $j=1, 2$. Take any $\lambda \in \mathbb{C} \setminus U_1$ and consider the operator $S := [(\lambda I - T)|_{X_1}]^{-1}$. Then we have $(\lambda I^* - T^*) S^* |_{Y_1} = I^* |_{Y_1}$, which implies $\lambda \in \rho(T^* |_{Y_1})$. Hence $\sigma(T^* |_{Y_1}) \subseteq U_1$. By the same the arguments, we conclude that $\sigma(T^* |_{Y_2}) \subseteq U_2$. By Theorem 2.3, T^* is (E)-super-decomposable. This completes the proof.

In Theorem 2.5, the opposite implications holds in reflexive Banach space. We have the following.

COROLLARY 2.6. *Let X be a reflexive Banach space. Then $T \in \mathcal{L}(X)$ is super-decomposable if and only if T^* is.*

Proof. If T is a super-decomposable operator on a reflexive Banach space X , then by Theorem 2.5, T^* is super-decomposable and so $(T^*)^*$ is super-decomposable. Since $T = T^{**}$, the proof is finished.

LEMMA 2.7 [8]. *Let $T \in \mathcal{L}(X)$ and let σ be a separate part of $\sigma(T)$. Let*

$$\mathcal{E}(\sigma, T) := \frac{1}{2\pi i} \int_{\Gamma} (\lambda - T)^{-1} d\lambda$$

be the spectral projection corresponding to σ , where Γ is a system of closed Jordan curves situated in $\rho(T)$, surrounding σ and separating the sets σ and $\sigma' = \sigma(T) \setminus \sigma$. Then $\mathcal{E}(\sigma, T)X$ is a spectral maximal space of T and $\sigma(T|_{\mathcal{E}(\sigma, T)X}) = \sigma$.

THEOREM 2.8. *If the spectrum $\sigma(T)$ of $T \in \mathcal{L}(X)$ is totally disconnected, then T is (E)-super-decomposable.*

Proof. Let $\{U, V\}$ be any open cover of $\sigma(T)$. We put $\sigma = \sigma(T) \cap U$. We first take the case that σ is a separate part of $\sigma(T)$. Then $\delta := \sigma(T) \setminus \sigma$ is also a separate part of $\sigma(T)$. Thus both $\mathcal{E}(\sigma, T)$ and $\mathcal{E}(\delta, T)$ are defined and $\mathcal{E}(\sigma, T)^2 = \mathcal{E}(\sigma, T)$. We claim that the operator $R := \mathcal{E}(\sigma, T)$ satisfies the conditions of Definition 2.2 for $T \in \mathcal{L}(X)$.

Clearly, $\sigma(T|R(X)) = \sigma(T|\mathcal{E}(\sigma, T)X) = \sigma \subseteq U$. By the functional calculus, we have

$$T = \frac{1}{2\pi i} \int_{\Gamma} \lambda(\lambda - T)^{-1} d\lambda, \quad I = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - T)^{-1} d\lambda$$

where Γ is a simple closed contour surrounding $\sigma(T)$ and lying in $\rho(T)$. An elementary line integral shows that $\mathcal{E}(\sigma, T) + \mathcal{E}(\delta, T) = \mathcal{E}(\sigma(T), T) = I$. Thus $I - \mathcal{E}(\sigma, T) = \mathcal{E}(\delta, T)$. Hence $\sigma(T|(I - R)(X)) = \sigma(T|\mathcal{E}(\delta, T)X) = \delta \subseteq V$. Clearly, $RT = TR$. It follows that T is (E)-super-decomposable.

We now consider the case that $\sigma(\neq \phi)$ is not a separate part of $\sigma(T)$. This can be happen when $\sigma(T)$ is an infinite set; (for example, $z_0 \in \sigma(T)$ is a limit point of a sequence $\{z_n\}$ in $\sigma(T)$ such that z_0 is a boundary point of U and $U \cap \sigma(T) = \{z_n\}$, then $\{z_n\}$ is not a separate part of $\sigma(T)$). Since $\sigma(T)$ is totally disconnected, there exists a separate part δ such that $\delta \subset \sigma, \delta \neq \sigma, \delta \cap V \neq \phi$ and $\sigma(T) \setminus \delta \subseteq V$. Thus the projection operator $\mathcal{E}(\delta, T)$ is defined. By the same reasoning in the proof of first case, we conclude that $\sigma(T|\mathcal{E}(\delta, T)X) = \delta \subseteq U, \sigma(T|(I - \mathcal{E}(\delta, T))X) = \sigma(T) \setminus \delta \subseteq V$ and $T\mathcal{E}(\delta, T) = \mathcal{E}(\delta, T)T$. This completes the proof.

The following Corollary is an immediate consequence of the Theorem 2.8

COROLLARY 2.9. *Any compact operators and quasi-nilpotent operators are (E)-super-decomposable.*

REMARK 2.10 [10]. *Let \mathcal{A} be a Banach algebra with unit 1, and let X be a complex Banach space. Assume that $\Theta : \mathcal{A} \rightarrow \mathcal{L}(X)$ is a continuous Banach representation of \mathcal{A} on X such that $\Theta(1) = I$. Then $\sigma_{\mathcal{A}}(x) = \sigma_{\mathcal{L}(X)}(\Theta(x))$.*

From Theorem 2.8, we have the following immediate consequence.

COROLLARY 2.11. *Let $\Theta : \mathcal{A} \rightarrow \mathcal{L}(X)$ be a continuous Banach representation of a Banach algebra \mathcal{A} on a Banach space X such that $\Theta(1) = I$. If the spectrum $\sigma(x)(x \in \mathcal{A})$ is totally disconnected, then $\Theta(x)$ is (E)-super-decomposable.*

We recall that a subspace $Y \subseteq X$ is said to be *hyperinvariant* if it is invariant under every operators in $\mathcal{L}(X)$ which commutes with T .

LEMMA 2.12. Let $T \in \mathcal{L}(X)$ be (E)-super-decomposable, that is, for every pair of open cover $\{U, V\}$ of \mathbb{C} , there exists $R \in \mathcal{L}(X)$ such that $R^2 = R$, $RT = TR$, $\sigma(T|R(X)) \subseteq U$ and $\sigma(T|(I - R)(X)) \subseteq V$. If $Y \in AI(T)$ is hyperinvariant, then $R(Y) \in AI(T)$.

Proof. Let $f : V_f \rightarrow X$ be analytic on a region V_f such that $(\lambda - T)f(\lambda) \in R(Y)$ for $\lambda \in V_f$. Then $(\lambda - T)(I - R)f(\lambda) \in (I - R)R(Y) = \{0\}$. Thus $(\lambda - T)(I - R)f(\lambda) = 0$ for $\lambda \in V_f$. By the SVEP of T , $(I - R)f(\lambda) = 0$ for $\lambda \in V_f$, and so $f(\lambda) = Rf(\lambda)$. Since $(\lambda - T)f(\lambda) = R(\lambda - T)f(\lambda) \in R(Y) \subseteq Y$ and $Y \in AI(T)$, we have $f(\lambda) \in Y$. Thus $f(\lambda) = Rf(\lambda) \in R(Y)$ for $\lambda \in V_f$. Hence $R(Y)$ is analytically invariant under T .

PROPOSITION 2.13 [10]. Let $T \in \mathcal{L}(X)$, and let $Y, Z \in Lat(T)$ with $Y \subset Z$. The following properties hold.

- (1) If $Y \in AI(T)$, then $Y \in AI(T|Z)$.
- (2) The quotient space $Z/Y \in AI(T^Y)$ if and only if $Z \in AI(T)$.

THEOREM 2.14. Let $T \in \mathcal{L}(X)$ be (E)-super-decomposable. For any hyperinvariant $Y \in AI(T)$, $T|Y$ is (E)-super-decomposable.

Proof. We put $T_Y = T|Y$. For every pair of open sets $U, V \subseteq \mathbb{C}$ such that $U \cup V = \mathbb{C}$, there exists some $R \in \mathcal{L}(X)$ such that $R^2 = R$, $RT = TR$, $\sigma(T|R(X)) \subseteq U$ and $\sigma(T|(I - R)(X)) \subseteq V$. By Lemma 2.12, $R(Y) \in AI(T)$. Thus $\sigma(T_Y|R(Y)) = \sigma(T|Y \cap R(Y)) = \sigma(T|R(Y)) = \sigma(T|R(Y) \cap R(X))$. Since $R(X) \in AI(T)$ and $R(Y) \cap R(X) \subseteq R(X)$, we have $R(X) \cap R(Y) \in AI(T|R(X))$. Consequently,

$$\sigma(T_Y|R(Y)) = \sigma(T|Y \cap R(Y)) \subseteq \sigma(T|R(X)) \subseteq U.$$

Clearly, $T_Y R = R T_Y$. Similarly, we obtain $\sigma(T_Y|(I - R)(X)) \subseteq V$. We have proved that T_Y is (E)-super-decomposable.

Recall from [10] that every spectral maximal space is hyperinvariant space.

We say that T is *strongly decomposable* if for every $Y \in SM(T)$, $T|Y \in \mathcal{L}(Y)$ is decomposable. The following result is an immediate consequence of the Theorem 2.14.

COROLLARY 2.15. If $T \in \mathcal{L}(X)$ is (E)-super-decomposable, then both T and T^* are strongly decomposable.

COROLLARY 2.16. Let $T \in \mathcal{L}(X)$ be (E)-super-decomposable. For any $Y \in SM(T)$, $T|_Y$ is (E)-super-decomposable.

For an operator $T \in \mathcal{L}(X)$, if $A \subseteq \mathbb{C}$ then the maximal algebraic spectral subspace $E_T(A)$ is the largest linear subspace of X such that $(T - \lambda)E_T(A) = E_T(A)$ holds for all $\lambda \in \mathbb{C} \setminus A$.

There is a reducing hyperinvariant subspace for $T \in \mathcal{L}(H)$ on a Hilbert space H .

EXAMPLE 2.17. Let $T \in \mathcal{L}(H)$ be such that T is one to one. Consider linear subspaces

$$H_0 := \bigcap_{n \geq 0} T^n H, \quad H_1 := \bigoplus_{n \geq 0} T^n (\text{Ker} T^*).$$

By the Wold decomposition supplies us $H = H_0 \oplus H_1$ and H_0 is reducing for T . More precisely that $TH_0 = H_0$, $TH_1 \subset H_1$. Thus there is a projector P such that $PH = H_0$ and $PT = TP$. Next, we prove that

$$E_T(\mathbb{C} \setminus \{0\}) = H_0 = \bigcap_{n \geq 0} T^n H.$$

Since $TH_0 = H_0$, $H_0 \subseteq E_T(\mathbb{C} \setminus \{0\})$ by maximality. Since T is one to one, it follows from ([15], Lemma) that

$$E_T(\mathbb{C} \setminus \{0\}) = \bigcap_{n=1}^{\infty} T^n H = \bigcap_{n \geq 0} T^n H = H_0.$$

Moreover, $E_T(\mathbb{C} \setminus \{0\})$ is a hyperinvariant subspace for T . Hence $H_0 = \bigcap_{n \geq 0} T^n H$ is a reducing hyperinvariant subspace of H .

THEOREM 2.18. Let $T \in \mathcal{L}(H)$ be (E)-super-decomposable on a complex Hilbert space H . Then both $T|_Y$ and $T^*|_Y$ are (E)-super-decomposable for any reducing hyperinvariant subspace Y of H .

Proof. Let Y be a reducing hyperinvariant subspace of H . Then there is a projection operator P such that $PH = Y$ and $PT = TP$. It

follows ([10], Proposition 2.14) that $PH = Y$ is analytically invariant under T . For any open cover $\{U, V\}$ of \mathbf{C} , there is $R \in \mathcal{L}(H)$ such that $R^2 = R, TR = TR, \sigma(T|RH) \subseteq U$ and $\sigma(T|(I - R)H) \subseteq V$. Since Y is hyperinvariant, RY is closed in H . We want to show that $RY \in AI(T)$. Let $f : V_f \rightarrow H$ be any analytic function on some open $V_f \subset \mathbf{C}$ and satisfy condition $(T - \lambda I)f(\lambda) \in RY$ on V_f . Then $f(\lambda) \in Y$ for all $\lambda \in V_f$, because $Y = PH \in AI(T)$. Since

$$(T - \lambda I)(I - R)f(\lambda) \in (I - R)RY = \{0\},$$

it follows from SVEP of T that $f(\lambda) = Rf(\lambda)$. Hence $f(\lambda) = Rf(\lambda) \in RY$ for all $\lambda \in V_f$, this implies RY is an analytically invariant subspace of T . Clearly, $(T|Y)R = R(T|Y)$, $\sigma((T|Y)|RH) = \sigma(T|Y \cap RH) \subseteq \sigma(T|RH) \subseteq U$ and $\sigma((T|Y)|(I - R)H) = \sigma(T|Y \cap (I - R)H) \subseteq \sigma(T|(I - R)H) \subseteq V$. Therefore $T|Y$ is (E)-super-decomposable by Theorem 2.14. On the other hand, it is well known that Y reduces T if and only if $PT = TP$ if and only if Y reduces T^* . By the same reasoning, we have $Y \in AI(T^*)$. Moreover, since $R^* = R = R^2$, $R^*Y = RY \subset Y$ and $RT^* = T^*R$. Hence $R^*Y \in AI(T^*)$ and the same argument shows that $T^*|Y$ is (E)-super-decomposable.

It is known that if $T_i \in \mathcal{L}(X_i)$ and $Y_i \in Lat(T_i)$ for $i = 1, 2$. then $\sigma(T_1 \oplus T_2|Y_1 \oplus Y_2) = \sigma(T_1|Y_1) \cup \sigma(T_2|Y_2)$, where $Y_1 \oplus Y_2$ is considered as subspace of $X_1 \oplus X_2 = \{x_1 \oplus x_2 = (x_1, x_2) \mid x_i \in X_i, i = 1, 2\}$ and $\|x_1 \oplus x_2\| = (\|x_1\|^2 + \|x_2\|^2)^{\frac{1}{2}}$.

THEOREM 2.19. *If $T_j \in \mathcal{L}(X_j)(j=1,2)$ are (E)-super-decomposable, then $T = T_1 \oplus T_2 \in \mathcal{L}(X_1 \oplus X_2)$ is also (E)-super-decomposable.*

Proof. Assume that $T_j \in \mathcal{L}(X_j)$ are (E)-super-decomposable for $j = 1, 2$. Then for every pair of open sets $U, V \subseteq \mathbf{C}$ such that $U \cup V = \mathbf{C}$, there exists some $R_j \in \mathcal{L}(X_j)$ such that $R_j^2 = R_j, R_j T_j = T_j R_j, \sigma(T_j|R_j(X_j)) \subseteq U$ and $\sigma(T_j|(I - R_j)(X_j)) \subseteq V$ for $j = 1, 2$. Let $R := R_1 \oplus R_2 \in \mathcal{L}(X_1 \oplus X_2)$. Clearly, $R^2 = R$ and $RT = TR$. Since $\sigma(T|R(X_1 \oplus X_2)) = \sigma(T_1|R_1(X_1) \oplus T_2|R_2(X_2)) = \sigma(T_1|R_1(X_1)) \cup \sigma(T_2|R_2(X_2)) \subseteq U$. A similar argument ensures that $\sigma(T|((I_1 \oplus I_2) - R)(X_1 \oplus X_2)) \subseteq V$. Hence T is (E)-super-decomposable.

THEOREM 2.20. *Let $T = T_1 \oplus T_2 \in \mathcal{L}(X_1 \oplus X_2)$ be a (E)-super-decomposable. If both X_1 and X_2 are hyperinvariant under T , then both T_1 and T_2 are (E)-super-decomposable.*

Proof. Assume that T is (E)-super-decomposable. We claim that $X_j \in AI(T)$ for $j = 1, 2$. Let $f : V_f \rightarrow X_1 \oplus X_2$ be analytic function on a region V_f such that $(\lambda - T)f(\lambda) \in X_1$ for $\lambda \in V_f$. Clearly, $f(\lambda) = f_1(\lambda) \oplus f_2(\lambda)$. Since $f_1(\lambda)$ and $f_2(\lambda)$ are analytic on V_f , we have $(\lambda - T)f(\lambda) = (\lambda I_1 - T_1)f_1(\lambda) \oplus (\lambda I_2 - T_2)f_2(\lambda) \in X_1 \equiv X_1 \oplus \{0\}$. Thus $(\lambda I_2 - T_2)f_2(\lambda) = 0$. Since T has the SVEP, T_2 has the SVEP. Thus $f_2(\lambda) = 0$ on V_f . Hence $f(\lambda) = f_1(\lambda) \in X_1$ on V_f , and so $X_1 \in AI(T)$. By Theorem 2.14, $T|_{X_1} = T_1$ is (E)-super-decomposable. A similar argument ensures that T_2 is also (E)-super-decomposable.

PROPOSITION 2.21. *If $T \in \mathcal{L}(X)$ is (E)-super-decomposable, then $L(T) \in \mathcal{L}(\mathcal{L}(X))$ is also (E)-super-decomposable, where $L(T) : \mathcal{L}(X) \rightarrow \mathcal{L}(X)$ is defined by $L(T)A = TA$ for $A \in \mathcal{L}(X)$.*

Proof. Given an arbitrary open covering $\{U, V\}$ of \mathbb{C} , there exists $R \in \mathcal{L}(X)$ such that $R^2 = R, RT = TR, \sigma(T|R(X)) \subseteq U$ and $\sigma(T|(I - R)X) \subseteq V$. Let $\tilde{R} := L(R)$. Clearly, $(\tilde{R})^2 = \tilde{R}$ and $\tilde{R}L(T) = L(T)\tilde{R}$. It remains to show that $\sigma(L(T)|\tilde{R}(\mathcal{L}(X))) \subseteq U$. Let $\lambda \in \mathbb{C} \setminus U$. Then there exists an inverse operator $S := [(T - \lambda)|R(X)]^{-1}$. We put $\tilde{S} := L(S)$. Then $(L(T) - \lambda)\tilde{S}L(R)A = \tilde{R}A$ for all $A \in \mathcal{L}(X)$, that is, $(L(T) - \lambda)\tilde{S}|\tilde{R}(\mathcal{L}(X)) = \tilde{I}|\tilde{R}(\mathcal{L}(X))$. Hence $\sigma(L(T)|\tilde{R}(\mathcal{L}(X))) \subseteq U$. By the same reasoning, $\sigma(L(T)|(I - \tilde{R})(\mathcal{L}(X))) \subseteq V$. Hence $L(T)$ is (E)-super-decomposable.

We turn to the study of the stability of (E)-super-decomposability under functional calculus.

PROPOSITION 2.22. *Let $T \in \mathcal{L}(X)$, and let $f : G \rightarrow \mathbb{C}$ be analytic on an open neighborhood G of $\sigma(T)$. If T is (E)-super-decomposable, then*

$$f(T) := \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda - T)^{-1} d\lambda$$

is (E)-super-decomposable, where Γ is a simple closed contour which is surrounding $\sigma(T)$ and lying in $\rho(T)$.

Proof. For any open cover $\{U, V\}$ of $\sigma(f(T))$ (or, \mathbf{C}), we have $\sigma(f(T)) = f(\sigma(T)) \subseteq U \cup V$ by spectral mapping theorem. Thus $\{f^{-1}(U), f^{-1}(V)\}$ is an open covering of $\sigma(T)$. By the assumption, there exists $R \in \mathcal{L}(X)$ such that $R^2 = R, RT = TR, \sigma(T|R(X)) \subseteq f^{-1}(U)$ and $\sigma(T|(I - R)(X)) \subseteq f^{-1}(V)$. we also obtain,

$$\begin{aligned} f(T)R &= \left(\frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda - T)^{-1} d\lambda \right) R \\ &= \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda - T)^{-1} R d\lambda \\ &= Rf(T). \end{aligned}$$

It suffices to show that $\sigma(f(T)|R(X)) \subseteq U$. Since $R(X)$ is ν -space under $f(T)$, $\sigma(f(T)|R(X)) = f(\sigma(T|R(X))) \subseteq U$. By the same method, we obtain $\sigma(T|(I - R)(X)) \subseteq V$. We have proved that $f(T)$ is (E)-super-decomposable.

It is evident that without some assumptions on f we cannot obtain the converse property to the preceding Proposition 2.21. Such an additional condition is given in the following.

PROPOSITION 2.23. *Let $T \in \mathcal{L}(X)$, and let $f : G \rightarrow \mathbf{C}$ be analytic and injective on an open neighborhood G of $\sigma(T)$. If $f(T)$ is (E)-super-decomposable, then so is T .*

Proof. If f is constant, then clearly, T is (E)-super-decomposable. Suppose that f is not constant. For every open covering $\{U, V\}$ of \mathbf{C} , we have $\{f(U), f(V)\}$ is an open covering of $\sigma(f(T)) = f(\sigma(T))$, by open mapping theorem and spectral mapping theorem. By the assumption, there exists $R \in \mathcal{L}(X)$ such that $R^2 = R, Rf(T) = f(T)R, \sigma(f(T)|R(X)) \subseteq f(U)$ and $(f(T)|(I - R)(X)) \subseteq f(V)$. Thus $\sigma(f(T)|R(X)) = f(\sigma(T|R(X))) \subseteq f(U)$. Hence $\sigma(T|R(X)) \subseteq U$, since f is injective. A similar argument ensures that $\sigma(T|(I - R)(X)) \subseteq V$. Hence T is (E)-super-decomposable.

As an immediate consequence of Proposition 2.22 and Proposition 2.23 one obtains.

COROLLARY 2.24. Let $T \in \mathcal{L}(X)$, and let $f : G \rightarrow \mathbb{C}$ be analytic and injective on an open neighborhood G of $\sigma(T)$. Then T is (E)-super-decomposable if and only if so is $f(T)$.

THEOREM 2.25. Let $T \in \mathcal{L}(X)$ be (E)-super-decomposable. Then the coinduced operator $T^Y \in \mathcal{L}(X/Y)$ is (E)-super-decomposable for any $Y \in \text{Lat}(T)$.

Proof. Let $Y \in \text{Lat}(T)$. For any open covering $\{U, V\}$ of \mathbb{C} , there exists $R \in \mathcal{L}(X)$ such that $R^2 = R$, $RT = TR$, $\sigma(T|R(X)) \subseteq U$ and $\sigma(T|(I - R)(X)) \subseteq V$. Clearly, $R^Y T^Y = T^Y R^Y$, $(R^Y)^2 = R^Y$ and $R^Y(X/Y) = \{[Rx] : x \in X\} = R(X)/Y$. It suffices to show that $\sigma(T^Y|R^Y(X/Y)) = \sigma(T^Y|R(X)/Y) \subseteq \sigma(T|R(X))$. For any $\lambda \in \rho(T|R(X))$, there exists an inverse operator $S = ((\lambda - T)|R(X))^{-1}$ such that $S(\lambda - T)|R(X) = (\lambda - T)S|R(X) = I|R(X)$. Thus $S(\lambda - T)Rx = (\lambda - T)SRx = Rx$ for all $x \in X$ and hence $(S(\lambda - T))^Y([Rx]) = ((\lambda - T)S)^Y([Rx]) = [Rx]$ for every $x \in X$. This implies

$$S^Y(\lambda - T)^Y|R(X)/Y = (\lambda - T)^Y S^Y|R(X)/Y = I|R(X)/Y.$$

And so $S^Y = ((\lambda - T)^Y)^{-1} = (\lambda I^Y - T^Y)^{-1}$ holds on $R(X)/Y$. It follows that $\sigma(T^Y|R^Y(X/Y)) \subseteq \sigma(T|R(X)) \subseteq U$. Similarly, we have $\sigma(T^Y|(I^Y - R^Y)(X/Y)) = \sigma(T^Y|(I - R)X/Y) \subseteq \sigma(T|(I - R)(X)) \subseteq V$. Hence T^Y is (E)-super-decomposable.

COROLLARY 2.26. The following statements are equivalent:

- (1) $T \in \mathcal{L}(X)$ is (E)-super-decomposable.
- (2) $T|Y \in \mathcal{L}(Y)$ is (E)-super-decomposable for any hyperinvariant subspace $Y \in \text{AI}(T)$.
- (3) $T^Y \in \mathcal{L}(X/Y)$ is (E)-super-decomposable for any $Y \in \text{Lat}(T)$.

Proof. The implications (1) \implies (2) and (1) \implies (3) follow from Theorem 2.14 and Theorem 2.25.

(2) \implies (1). Clearly, X is a trivial T -analytically invariant subspace. Hence, from Theorem 2.14, T is (E)-super-decomposable.

(3) \implies (1). Clearly, $\{0\} \in \text{Lat}(T)$. Hence it follows from Theorem 2.25 that $T \cong T^{\{0\}}$ is (E)-super-decomposable.

An immediate consequence of Corollary 2.26 is following.

COROLLARY 2.27. *Let $T \in \mathcal{L}(X)$ be (E)-super-decomposable.*

(a) *If $Y_1 \in AI(T)$ is analytically hyperinvariant under T , $Y_2 \subseteq Y_1$ and $Y_2 \in Lat(T|Y_1)$, then $(T|Y_1)^{Y_2}$ is (E)-super-decomposable.*

(b) *If $Y_1 \in Lat(T)$, $Z \subseteq X/Y_1$ and $Z \in AI(T^{Y_1})$ is analytically hyperinvariant under T^{Y_1} , then $T^{Y_1}|Z$ is (E)-super-decomposable.*

3. Examples of (E)-super-decomposable operators

In this section, we present examples of (E)-super-decomposable operators. This class, (E)-super-decomposable operator, contain many interesting examples; nilpotent, finite dimensional and Hilbert-Schmidt operators by Theorem 2.8. Moreover, well-bounded operator defined on a reflexive Banach space is also (E)-super-decomposable.

Consider a complex Banach space $L^p(X, \mathcal{A}, \mu)$ for $1 \leq p \leq \infty$. Let $\phi \in L^\infty(X, \mathcal{A}, \mu)$, where (X, \mathcal{A}, μ) is a finite measure space. The multiplication operator corresponding to ϕ is the bounded operator M_ϕ on $L^p(X, \mathcal{A}, \mu)$ defined by $(M_\phi f)(x) = \phi(x)f(x)$ for all $f \in L^p(X, \mathcal{A}, \mu)$ ($1 \leq p \leq \infty$) and $x \in X$.

EXAMPLE 1. *Every spectral operator T is (E)-super-decomposable. In particular, the multiplication operator M_ϕ defined on $L^p(X, \mathcal{A}, \mu)$ is (E)-super-decomposable for every $\phi \in L^\infty(X, \mathcal{A}, \mu)$.*

Proof. Let $\mathcal{E}(\cdot)$ be the spectral measure for T . Then $T\mathcal{E}(\sigma) = \mathcal{E}(\sigma)T$ and $\sigma(T|\mathcal{E}(\sigma)(X)) \subseteq \bar{\sigma}$ for all $\sigma \in \beta(\mathbb{C})$, where $\beta(\mathbb{C})$ is the σ -algebra of Borel subsets of \mathbb{C} . For every pair of open sets $U, V \subseteq \mathbb{C}$ such that $U \cup V = \mathbb{C}$, we choose open sets $U_1, V_1 \subseteq \mathbb{C}$ such that $U_1 \subseteq \bar{U}_1 \subseteq U, V_1 \subseteq \bar{V}_1 \subseteq V$ and $U_1 \cup V_1 = \mathbb{C}$. We claim that the operator $R := \mathcal{E}(U_1)$ satisfies the conditions of Definition 2.2 for $T \in \mathcal{L}(X)$. Clearly, $RT = TR$, $R^2 = R$ and $\sigma(T|R(X)) = \sigma(T|\mathcal{E}(U_1)X) \subseteq \bar{U}_1 \subseteq U$. Similarly, we have $\sigma(T|(I - R)(X)) \subseteq V$. Hence T is (E)-super-decomposable. Finally, for $S \in \beta(\mathbb{C})$, we define $\mathcal{E}(S) = M_{\chi_S} \circ \phi$, where χ_S denote the characteristic function corresponding to a set S . It follows from ([12], Proposition 2.1) that $\mathcal{E}(\cdot)$ is a spectral measure which makes M_ϕ a spectral operator.

Let \mathcal{A} be a Banach algebra with unity 1. Each x in \mathcal{A} determines a pair of linear mappings U_x, V_x on \mathcal{A} via the formulas $U_x y = xy$, $V_x y = yx$. Then U_x and V_x are in $\mathcal{L}(\mathcal{A})$.

EXAMPLE 2. *If the spectrum $\sigma_{\mathcal{A}}(x)$ of $x \in \mathcal{A}$ is totally disconnected, then both U_x and V_x are (E)-super-decomposable.*

Proof. The algebras $\mathcal{U} = \{U_x : x \in \mathcal{A}\}$ and $\mathcal{V} = \{V_x : x \in \mathcal{A}\}$ are full subalgebras of $\mathcal{L}(\mathcal{A})$, and $\sigma_{\mathcal{A}}(x) = \sigma_{\mathcal{L}(\mathcal{A})}(U_x) = \sigma_{\mathcal{L}(\mathcal{A})}(V_x)$. Since $\sigma_{\mathcal{A}}(x)$ is totally disconnected, $\sigma_{\mathcal{L}(\mathcal{A})}(U_x)$ and $\sigma_{\mathcal{L}(\mathcal{A})}(V_x)$ are totally disconnected. Whence the Theorem 2.8 is applied.

Let A denote a commutative complex Banach algebra. The set of all multiplicative linear functionals on A is denoted by $\mathcal{M}(A)$, the identically zero functional on A by ϕ_{∞} and $\mathcal{M}(A) \cup \{\phi_{\infty}\}$ by $\widetilde{\mathcal{M}}(A)$. Then $\mathcal{M}(A)$ and $\widetilde{\mathcal{M}}(A)$ are subsets of A^* (the dual space of A). When $\mathcal{M}(A)$ (or, $\widetilde{\mathcal{M}}(A)$) is equipped with the relative weak* topology, we call it the *spectrum* (or, *character space*) of A .

It is well known [6] that the spectrum $\mathcal{M}(A)$ is a locally compact Hausdorff space with one point compactification $\widetilde{\mathcal{M}}(A)$. In particular, if A has a unit element, then $\mathcal{M}(A)$ is compact.

EXAMPLE 3. *Let A be a commutative Banach algebra over \mathbb{C} and assume that the spectrum $\mathcal{M}(A)$ is totally disconnected. Then for every $a \in A$ and every algebraic homomorphism $\Phi : A \rightarrow \mathcal{L}(X)$, the operator $\Phi(a) \in \mathcal{L}(X)$ is (E)-super-decomposable.*

Proof. Suppose that $\mathcal{M}(A)$ is totally disconnected. Assume that A has no identity element. Consider the unitization $\widetilde{A} := A \oplus \mathbb{C}$ of the Banach algebra A and the canonical extension $\widetilde{\Phi} : \widetilde{A} \rightarrow \mathcal{L}(X)$ of the homomorphism Φ given by $\widetilde{\Phi}(x + \lambda) = \Phi(x) + \lambda I$ for all $x \in A$ and $\lambda \in \mathbb{C}$. Then $\widetilde{\mathcal{M}}(A)$ is the one point compactification of the locally compact space $\mathcal{M}(A)$ with the Gelfand topology. Since an operator $T \in \mathcal{L}(X)$ is (E)-super-decomposable if and only if $T - \mu I$ is (E)-super-decomposable for some $\mu \in \mathbb{C}$, it suffices to prove the assertion that for an operator $\widetilde{\Phi}(a) \in \mathcal{L}(X)$, for $a \in \widetilde{A}$ satisfies $0 \notin \sigma(a) = \widehat{a}(\mathcal{M}(A))$, where $\widehat{a} : \widetilde{\mathcal{M}}(A) \rightarrow \mathbb{C}$ denotes the Gelfand transform of a on $\widetilde{\mathcal{M}}(A)$. Now, given open sets $U, V \subseteq \mathbb{C}$ such that $U \cup V = \mathbb{C}$, we may assume that $\mu \in V$, where μ is the complex number for which

$a - \mu 1 \in A$. Then $K := \widetilde{\mathcal{M}}(A) \setminus \widehat{a}^{-1}(V)$ is a compact subset of $\mathcal{M}(A)$ and $\widehat{a}^{-1}(U) \cap \mathcal{M}(A)$ is an open neighborhood of K in $\mathcal{M}(A)$. Since $\mathcal{M}(A)$ is locally compact, there exists a compact neighborhood L of K in $\mathcal{M}(A)$ such that $L \subseteq \widehat{a}^{-1}(U)$. Since L is compact and totally disconnected, there is a compact and open subset O of $\mathcal{M}(A)$ such that $K \subseteq O \subseteq L \subseteq \widehat{a}^{-1}(U)$. Since O and $\widetilde{\mathcal{M}}(A) \setminus O$ are disjoint non-empty compact sets, \widetilde{A} contains an idempotent e with $\widehat{e} = 1$ on O and $\widehat{e} = 0$ on $\widetilde{\mathcal{M}}(A) \setminus O$ by the Shilov idempotent theorem. We claim that the operator $R := \widetilde{\Phi}(e) \in \mathcal{L}(X)$ satisfies the conditions of Definition 2.2 for $T := \widetilde{\Phi}(a)$. Clearly, $R^2 = R$ and $RT = TR$. It remains to show that $\sigma(T|R(X)) \subseteq U$ and $\sigma(T|(I - R)(X)) \subseteq V$. Given any $\lambda \in \mathbb{C} \setminus U$, since $\sigma(ae) = \widehat{a\widehat{e}}(\widetilde{\mathcal{M}}(A)) \subseteq U \cup \{0\}$, there exists some $e_\lambda \in \widetilde{A}$ satisfying $(ae - \lambda)e_\lambda = e = e_\lambda(ae - \lambda)$, whenever $\lambda \neq 0$. Since $a \in \widetilde{A}$ is invertible and $e \in \widetilde{A}$ is idempotent, we have $(a - \lambda)e_\lambda e = e^2 = e$ for some $e_\lambda \in \widetilde{A}$ including the case $\lambda = 0$. It follows that $(T - \lambda I)\widetilde{\Phi}(e_\lambda e)x = \widetilde{\Phi}(e_\lambda e)(T - \lambda I)x = \widetilde{\Phi}(e)x = x$ and $\widetilde{\Phi}(e)x \in R(X)$ for all $x \in R(X)$, which implies $\sigma(T|R(X)) \subseteq U$. By the same reasoning, we have $\sigma(T|(I - R)(X)) \subseteq V$. Hence $T = \widetilde{\Phi}(a)$ is (E)-super-decomposable.

References

1. E. Albrecht, *On decomposable operators*, Integral Equations operator theory 2 (1979), 1 -10.
2. E. Albrecht and M.M. Neumann, *Automatic continuity for operators of local type*, in *Radical Banach algebras and Automatic Continuity*, Springer Lecture Notes in Math 975., (1983), 342 -355.
3. E. Albrecht, J. Eschmeier and M.M. Neumann, *Some topics in the theory of decomposable operators*, in *Advances in invariant subspaces and other results of operator theory*, Birkhauser Verlag, Basel (1986), 15 -34.
4. C. Apostol, *Spectral decompositions and functional calculus*, Rev. Roum. Math. Pures et Appl., 13 (1968), 1481 -1528.
5. R.G.Bartle and C.A.Kariotis, *Some Localizations of the Spectral Mapping Theorem*, Duke Math. J. 40 (1973), 651 -660.
6. S.K. Berberian, Springer, New York (1974).
7. E. Berkson, *A Fourier Analysis theory of abstract spectral decompositions*, in *Surveys of some recent results in operator theory*, Longman Scientific and Technical 1 (1988), 95 -132.
8. I.Colojoara and C.Foias, *Theory of generalized spectral operators*, Gordon and Breach, New York (1968).

9. H.R. Dowson, *Spectral theory of linear operators*, Academic Press, London-New York-San Francisco (1978).
10. I. Erdelyi and R. Lange, *Spectral Decompositions on Banach spaces*, Springer Lecture Notes 623 (1977).
11. I. Erdelyi and Wang Sheng Wang, *On strongly decomposable operators*, Pacific J. Math., 110 (1984), 287 -296.
12. C.K. Fong and L. Lam, *On spectral theory and convexity*, Trans. Amer. Soc., 11 (1981), 59 -75.
13. S.H. Huang, *Each hyperinvariant subspace for multiplication operator is spectral*, Proc. Amer. Math. Soc., 106 (1989), 1057 -1061.
14. R. Lange, *Strongly analytic subspaces in operator theory and functional analysis*, edited by I. Erdelyi, *Research Notes in Mathematics 38*, Pitman, San Francisco (1979).
15. K.B. Laursen, *Some remarks on the surjectivity spectrum of linear operators*, Czechoslovak Math J., 39 (1989), 730 -739.
16. K.B. Laursen and M.M. Neumann, *Decomposable operators and automatic continuity*, J. Operator Theory 15 (1986), 33 -51.
17. K.B. Laursen, *Intertwining with isometries*, Conference on automatic continuity and Banach algebras Canberra (1989), 254 - 259.

Department of Mathematics,
Sogang University,
Seoul 121-742, Korea