C-TOLERANCE STABILITY
OF DYNAMICAL SYSTEMS

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1. Introduction.

Zeeman[8] introduced the concepts of tolerance stability of dynamical systems on a compact metric space. In this paper we investigate tolerance stability of dynamical systems using the concepts of chain recurrence. And we get an equivalence condition for a dynamical system to be tolerance stable. Also we introduce the notion of C-tolerance stability using the concept of chain recurrence and it is shown that an equivalence condition for a dynamical system to be C-tolerance stable. We show that C-tolerance stability is invariant under conjugacy. Finally we give a necessary condition for which the notion of tolerance stability is equal to that of C-tolerance stability.

We consider homeomorphisms (or dynamical systems) acting on a compact metric space. Let $X$ denote a compact metric space with a metric $d$, and let $H(X)$ be the collection of all homeomorphisms of $X$ to itself topologized by the $C^0$-metric:

$$d_0(f, g) = \sup \{ d(f(x), g(x)) \mid x \in X \},$$

where $f$ and $g$ are elements in $H(X)$.

We say that a dynamical system $f \in H(X)$ is topologically stable if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $d_0(f, g) < \delta$, $g \in H(X)$, then there is a continuous surjection $h : X \to X$ with $fh = hg$ and $d_0(h, I_X) < \epsilon$, where $I_X : X \to X$ stands for the identity homeomorphism. We introduce the concept of tolerance stability for homeomorphisms which is weaker than that of topologically stability.

Let $K(X)$ be the set of all nonempty closed subsets of $X$ with the Hausdorff metric $\rho$: for any $A, B \in K(X)$,

$$\rho(A, B) = \max \{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \},$$

93
where \( d(a, B) = \inf \{ d(a, b) \mid b \in B \} \). Then the set \( K(X) \) with the metric \( \rho \) is again a compact metric space. Let \( K(K(X)) \) be the set of all nonempty closed subsets of \( K(X) \) with the Hausdorff metric \( \bar{\rho} \).

For any \( f \in H(X) \) and \( x \in X \), the set

\[
O(f, x) = \{ f^n(x) : n \in \mathbb{Z} \}
\]

is called the \( f \)-orbit closure of \( x \). Since the set \( O(f, x) \) can be interpreted as a point in \( K(X) \), we can consider the closure of the set \( \{ O(f, x) : x \in X \} \) in \( K(X) \), which is denoted by \( O(f) \). The set \( O(f) \) also may be interpreted as a point of \( K(K(X)) \). Hence we can consider the map \( O : H(X) \to K(K(X)) \) sending \( f \in H(X) \) to \( O(f) \).

2. Tolerance Stability.

We say that \( f \in H(X) \) is tolerance stable if the map \( O : H(X) \to K(K(X)) \), which assigns to each \( g \in H(X) \) the point \( O(g) \in K(K(X)) \), is continuous at \( f \) as we know in [1]. Suppose that \( X \) and \( Y \) are metric spaces with \( Y \) compact. A map \( h : X \to K(Y) \) is said to be upper (or lower) semi-continuous at \( x \in X \) if for any \( \epsilon > 0 \) there exists a neighborhood \( U \) of \( x \) such that for any \( z \in U \) we have

\[
h(z) \subset B_\epsilon(h(x)) \quad \text{(or} \quad h(x) \subset B_\epsilon(h(z)),
\]

respectively, where \( B_\epsilon(A) = \{ z \in X : d(x, z) < \epsilon \} \) for some \( x \in A \).

A map \( h : X \to K(Y) \) is continuous at \( x \in X \) if and only if \( h \) is upper and lower semicontinuous at \( x \in X \). In the following theorem, we see that the notion of tolerance stability is characterized.

**Theorem 2.1.** \( f \in H(X) \) is tolerance stable if and only if for any \( \epsilon > 0 \) there exists \( \delta > 0 \) such that if \( d_0(f, g) < \delta \) with \( g \in H(X) \) then for any \( x \in X \) there are \( y, z \in X \) satisfying

\[
\rho(O(f, g), O(g, y)) < \epsilon \quad \text{and} \quad \rho(O(f, z), O(g, x)) < \epsilon.
\]

**Proof.** Let \( f \in H(X) \) be tolerance stable. Then for any \( \epsilon > 0 \) there exists \( \delta > 0 \) such that if \( d_0(f, g) < \delta \) with \( g \in H(X) \), then \( O(g) \subset B_{\epsilon/2}(O(f)) \) and \( O(f) \subset B_{\epsilon/2}(O(g)) \). Let \( A \in O(g) \). Then there exists
$z \in X$ such that $\rho(A, O(g, z)) < \epsilon/2$. Since $A \in B_{\epsilon/2}(O(f))$, we have $\rho(A, O(f, x)) < \epsilon/2$ for any $x \in X$. Then we have $\rho(O(f, x), O(g, z)) < \epsilon$ for any $g \in B_{\delta}(f)$. Similarly we can show that if $d_0(f, g) < \delta$ and $x \in X$, then there exists $y \in X$ such that $\rho(O(g, x), O(f, y)) < \epsilon$.

Conversely, for any $\epsilon > 0$ there exists $\delta > 0$ such that if $d_0(f, g) < \delta$, $g \in H(X)$, then for any $x \in X$ there are $y, z \in X$ satisfying

$$\rho(O(g, y), O(f, x)) \leq \frac{\epsilon}{3} \quad \text{and} \quad \rho(O(f, z), O(g, x)) \leq \frac{\epsilon}{3}.$$  

Let $g \in B_{\delta}(f)$ and $A \in O(g)$. Then there exists $x \in X$ such that $\rho(A, O(g, x)) < \epsilon/2$. For $x \in X$, we select $z \in X$ satisfying

$$\rho(O(g, x), O(f, z)) \leq \frac{\epsilon}{3}.$$  

Then we obtain $\rho(A, O(f, z)) < \epsilon$. This means that $O(g) \subset B_{\epsilon/3}(O(f))$. Hence the map $O$ is upper semi-continuous at $f$. Similarly we can show that the map $O$ is lower semi-continuous at $f$. Consequently the map $f$ is tolerance stable  

Using the concept of chain recurrence, we will introduce the concept of $C$-tolerance stability of $f \in H(X)$, and show that the notion of $C$-tolerance stability is characterized. For our purpose we need some notations and definitions (see [7]).

Let $x$ and $y$ be two points in $X$, and let $\epsilon > 0$ be an arbitrary number. A finite sequence $\{x_i\}_{i=0}^n$ in $X$ is called an $\epsilon$-chain for $f$ from $x$ to $y$ if

1. $d(x_{i+1}, f(x_i)) < \epsilon$ for $i = 0, 1, \ldots, n-1$ and
2. $x_0 = x$ and $x_n = y$.

Using the concept, we define a relation "<" on $X$ induced by $f \in H(X)$ as follows: for any $x, y \in X$, $x < y$ if and only if for any $\epsilon > 0$ there exists an $\epsilon$-chain for $f$ from $x$ to $y$. As we know in [3], the $f$-chain orbit through $x \in X$, $C(f, x) = \{y \in X \mid x < y \text{ or } x > y \text{ or } x = y\}$, is compact in $X$ for each $x \in X$. Since each chain orbit $C(f, x)$ can be interpreted as a point in $K(X)$, we can consider the set $\{C(f, x) \mid x \in X\}$ in $K(X)$, which is denoted by $C(f)$. Then the set $C(f)$ is closed in $K(X)$. Hence the chain orbit map $C : H(X) \to K(K(X))$ sending $f$ to $C(f)$ is well-defined. We say that $f \in H(X)$ is $C$-tolerance stable if the map $C : H(X) \to K(K(X))$ is continuous at $f$. 

Theorem 2.2. $f \in H(X)$ is $C$-tolerance stable if and only if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $d_0(f, g) < \delta$ with $g \in H(X)$ then for any $x \in X$ there are $y, z \in X$ satisfying
\[
\rho(C(f, x), C(g, y)) < \epsilon \quad \text{and} \quad \rho(C(f, z), C(g, x)) < \epsilon.
\]
Proof. Similarly we can show that the theorem is true by the above theorem. \qed

Corollary 2.3. If $f \in H(X)$ is tolerance stable then it is $C$-tolerance stable.

Proof. It follows immediately from the fact that the orbit closure is contained in the chain orbit. \qed

We say that $f, g \in H(X)$ are topologically conjugate if there exists $h \in H(X)$ such that $hg = fh$. The $h \in H(X)$ is called a topological conjugacy between $f$ and $g$. In the following theorem, we see that $C$-tolerance stability is invariant under a topological conjugacy.

Theorem 2.4. Any homeomorphism which is topologically conjugate to a $C$-tolerance stable homeomorphism is also $C$-tolerance stable.

Proof. Let $f \in H(X)$ be $C$-tolerance stable, and suppose that $f, g \in H(X)$ are topologically conjugate. Let $h \in H(X)$ be a topological conjugacy between $f$ and $g$. Let $\epsilon > 0$ be arbitrary and choose $0 < \epsilon_1 < \epsilon$ such that if $d(a, b) < \epsilon_1$ then $d(h^{-1}(a), h^{-1}(b)) < \epsilon$ for $a, b \in X$. Applying Theorem 2.2, we shall complete the proof by showing that $g$ is $C$-tolerance stable. Since $f$ is $C$-tolerance stable, given $\epsilon_1 > 0$, there exists $\delta > 0$ such that if $d_0(f, f_0) < \delta$ then for any $x \in X$ there are $y, z \in X$ satisfying
\[
\rho(C(f_0, y), C(f, x)) < \epsilon_1 \quad \text{and} \quad \rho(C(f, z), C(f_0, x)) \epsilon_1.
\]
For the $\delta > 0$, choose $0 < \delta_1 < \delta$ such that if $d(a, b) < \delta_1$, $a, b \in X$, then $d(h(a), h(b)) < \delta$. Let $g_0 \in H(X)$ be such that $d_0(g, g_0) < \delta_1$, and let $f_0 = h g_0 h^{-1}$. Then we have
\[
d(h(g(x)), h(g_0(x))) = d(f(h(x)), f_0(h(x))) < \delta
\]
for any \( x \in X \), and so \( d_0(f, f_0) < \delta \). Then for any \( x \in X \), there exists \( h(y) \in X \) such that

\[
\rho(C(f_0, h(y)), C(f, h(x))) < \epsilon_1.
\]

Hence we have

\[
\rho(C(f_0, h(y)), C(f, h(x))) = \rho(C(hg_0 h^{-1}, h(y)), C(hg h^{-1}, h(x)))
\]

\[
= \rho(C(hg_0, y), C(hg, x))
\]

\[
< \epsilon_1.
\]

This means that given \( \epsilon > 0 \), there exists \( \delta_1 > 0 \) such that, for every \( x \in X \) there is \( y \in X \) satisfying

\[
\rho(C(g_0, y), C(g, x)) < \epsilon.
\]

Similarly, we can show that if \( d_0(g, g_0) < \delta_1 \), \( g_0 \in H(X) \), then for any \( x \in X \) there exists \( z \in X \) satisfying

\[
\rho(C(g, z), C(g_0, x)) < \epsilon.
\]

This completes the proof. \( \square \)

Finally we give a necessary condition to be \( C(f) = O(f) \), where \( f \in H(X) \). For this object, we need a lemma due to Z. Nitecki and M. Shub\[5\].

**Lemma 2.5.** Let \( M \) be a compact manifold of \( \dim \geq 2 \) with the metric \( d \), and let \( \epsilon > 0 \) be arbitrary. Then there exists \( \delta > 0 \) such that if \( \{(x_i, y_i) \in M \times M \mid i = 1, 2, \ldots, n\} \) is a finite set of points of \( M \times M \) satisfying

1. for each \( i = 1, 2, \ldots, n \), \( d(x_i, y_i) < \delta \) and
2. if \( i \neq j \), then \( x_i \neq x_j \) and \( y_i \neq y_j \),

then there exists \( h \in H(M) \) with \( d_0(h, 1_M) < \epsilon \) and \( h(x_i) = y_i \) for \( i = 1, 2, \ldots, n \).
Theorem 2.6. Let $M$ be a compact manifold of dim $\geq 2$. If $f \in H(M)$ is topologically stable, then we have $C(f) = O(f)$.

Proof. By definition, it is clear that $O(f) \subseteq C(f)$. Thus it is enough to show that $C(f, x) \subseteq O(f, x)$ for any $x \in M$. Let $d$ be the metric on $M$, and let $y \in C(f, x)$. Then we have $x < y$, or $x > y$, or $x = y$. Suppose that $x < y$, and let $k > 0$ be a positive integer. Since $f$ is topologically stable, given $1/k > 0$, there exists $\delta_1(k) > 0$ such that if $d_0(f, g) < \delta_1$ with $g \in H(M)$, then there is a continuous surjection $h : M \to M$ with $fh = hg$ and $d_0(h, I_M) < 1/k$.

Given $1/k > 0$, we choose $\delta_2(k) > 0$ satisfying the results of Lemma 2.5. Let $\{x_0, x_1, \ldots, x_{m_k}\}$ be a $\delta_2$-chain for $f$ from $x$ to $y$. Then the set $\{(f(x_0), x_1), \ldots, (f(x_{m_k-1}), x_{m_k})\}$ satisfies the hypothesis of Lemma 2.5. Hence there exists $\varphi \in H(M)$ such that

$$d_0(\varphi, I_M) < \frac{1}{k} \text{ and } \varphi(f(x_i)) = x_{i+1}$$

for $i = 0, 1, \ldots, m_k - 1$. By letting $g = \varphi f$, we get $d_0(f, g) < \delta_1$. Thus there is a continuous surjection $h$ with $fh = hg$, and we get

$$d(f^{m_k}(x), y) = d(f^{m_k}(x), g^{m_k}(x)) < \frac{m_k}{k}.$$  

This implies that $B_\epsilon(y) \cap O(f, x) \neq \emptyset$ for any $\epsilon > 0$, and so $y \in O(f, x)$. By now we have shown that if $x < y$ then $y \in O(f, x)$. Similarly we can show that if $x > y$ then $y \in O(f, x)$. This completes the proof. □

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