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# THE METRIC APPROXIMATION PROPERTY AND INTERSECTION PROPERTIES OF BALLS

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## 1. Introduction

In 1983 Harmand and Lima [5] proved that if $X$ is a Banach space for which $K(X)$, the space of compact linear operators on $X$, is an $M$-ideal in $L(X)$, the space of bounded linear operators on $X$, then it has the metric compact approximation property. A strong converse of the above result holds if $X$ is a closed subspace of either $\ell_p(1 < p < \infty)$ or $c_o$ [2, 15]. In 1979 J. Johnson [7] actually proved that if $X$ is a Banach space with the metric compact approximation property, then the annihilator $K(X)^\perp$ of $K(X)$ in $L(X)^*$ is the kernel of a norm-one projection in $L(X)^*$, which is the case if $K(X)$ is an $M$-ideal in $L(X)$.

Recently Lima [11] proved that the converse of Johnson's result is true for a Banach space $X$ with the Radon-Nikodym property. Combining earlier results he proved the following :

THEOREM 1.1 [11]. *Let $X$ be a Banach space with the Radon-Nikodym property. Then the following three statements are equivalent.*

(1) *$X$ has the metric compact approximation property.*
(2) *$K(X)^\perp$ is the kernel of a norm-one projection in $L(X)^*$.*
(3) *$K(X)$ has the $n.L(X).I.P.$ for all $n$.*

The purpose of this paper is to prove the equivalence of (1), (2) and (3) in the above theorem for a Banach space $X$ for which $K(X)$ is weakly Hahn-Banach smooth in $L(X)$ [Theorem 3.5]. The main part of the proof is that of implication (3) $\Longrightarrow$ (1). But this is a direct consequence of the following theorem which is a key theorem in this paper.

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THEOREM 1.2. *Let $X$ and $Y$ be Banach spaces. If $K(X,Y)$ is weakly Hahn-Banach smooth in $L(X,Y)$ and has the 3.$L(X,Y)$.I.P. then the closed unit ball $B_{K(X,Y)}$ of $K(X,Y)$ is dense in the closed unit ball $B_{L(X,Y)}$ of $L(X,Y)$ in the topology of uniform convergence on compact sets.*

In Theorem 3.3 we will also see that the above theorem is true if $K(X,Y)$ and $B_{K(X,Y)}$ are replaced by $\overline{F(X,Y)}$ and $B_{\overline{F(X,Y)}}$, respectively, where $F(X,Y)$ is the space of finite rank operators from $X$ to $Y$. As a consequence, if $\overline{F(X)}$ is weakly Hahn-Banach smooth in $L(X)$ and has the 3.$L(X)$.*I.P.*, then $X$ has the metric approximation property [Corollary 3.4].

Recall that a closed subspace $M$ of a Banach space $X$ is called an $M$-ideal in $X$ if $M^{\perp}$, the annihilator of $M$ in $X^*$, is the kernel of a projection $P$ in $X^*$ such that $\|x^*\| = \|Px^*\| + \|(I-P)x^*\|$ for all $x^* \in X^*$. Obviously $P$ has norm one. Alfsen and Effros [1] introduced the notion of an $M$-ideal and characterized an $M$-ideal by intersection properties of open balls. Later Lima [9] proved that a closed subspace $M$ of a Banach space $X$ is an $M$-ideal in $X$ if and only if $M$ has the $n$-intersection property ($n$.I.P.) for all $n = 3, 4, 5\cdots$, that is, if $\{B(a_i, r_i)\}_{i=1}^n$ is a family of closed balls in $X$ with nonempty intersection and $M \cap B(a_i, r_i) \neq \emptyset$ for all $i$, then we have

$$M \cap \bigcap_{i=1}^{n} B(a_i, r_i + \varepsilon) \neq \emptyset \qquad \text{for all } \varepsilon > 0.$$

A weaker intersection property is the n.X. intersection property. A closed subspace $M$ of a Banach space $X$ is said to have the $n$.X. intersection property ($n$.X.I.P.) if whenever $\{B(a_i, r_i)\}_{i=1}^n$ is a family of closed balls in $M$ such that

$$\bigcap_{i=1}^{n} B(a_i, r_i) \neq \emptyset \qquad \text{in } X$$

then

$$M \cap \bigcap_{i=1}^{n} B(a_i, r_i + \varepsilon) \neq \emptyset \qquad \text{for all } \varepsilon > 0.$$

Section 2 contains the background materials which will be used in Section 3. These materials taken from Lima's paper [10] mostly deal with the existence of a unique norm preserving Hahn-Banach extension and certain intersection properties of balls.

## 2. Preliminaries

If $X$ and $Y$ are Banach spaces $L(X,Y)$ (respectively, $K(X,Y)$) will denote the Banach space of all bounded linear operators (respectively, compact linear operators) from $X$ to $Y$ with the operator norm. Unless otherwise specified, the space $L(X,Y)$ will be equiped with the operator norm topology. If $X = Y$, then we will simply write $L(X)$ and $K(X)$ for $L(X,X)$ and $K(X,X)$, respectively. $F(X,Y)$ will denote the space of all finite rank operators from $X$ to $Y$. In general $F(X,Y)$ is not a closed subspace of $L(X,Y)$. The closure $\overline{F(X,Y)}$ of $F(X,Y)$ is $K(X,Y)$ if $Y$ has the approximation property, i.e. for every compact subset $K$ in $Y$ and every $\varepsilon > 0$, there exists a finite rank operator $T$ on $X$ such that $\|Tx - x\| < \varepsilon$ for all $x \in K$. If $X$ is a Banach space, $B(a,r)$ will denote the closed ball with the center $a$ and the radius $r$. We will write $B_X$ for $B(0,1)$. The dual space of a Banach space $X$ is denoted by $X^*$. If $M$ is a subspace of a Banach space $X$, $M^{\perp}$ will denote the annihilator of $M$ in $X^*$.

A closed subspace $M$ of a Banach space $X$ is said to be weakly Hahn-Banach smooth if every $f \in M^*$ that attains its norm on $B_M$ has a unique norm preserving extension to $X$. Such an $M$ is said to be Hahn-Banach smooth if every $f \in M^*$ has a unique norm preserving extension to $X$. The next two theorems due to Lima[10] relate unique norm preserving extensions of linear functionals and intersection properties of balls.

THEOREM 2.1 [10]. *The following statements are equivalent for a closed subspace $M$ of a Banach space $X$.*

(1) *$M$ is weakly Hahn-Banach smooth in $X$*
(2) *If $x \in M$, $y \in X$ with $\|x\| = 1 = \|y\|$ and $\varepsilon > 0$, then there exist $r \geq 1$ and $z \in M$ such that*

$$\max\{\|rx \pm (y - z)\|\} \leq r + \varepsilon.$$

THEOREM 2.2 [10]. *Let $M$ be a closed subspace of a Banach space $X$. For $n \geq 3$ the following statements are equivalent.*

(1) *$M$ has the n.X.I.P.*
(2) *If $f_1, f_2, \cdots, f_n \in M^*$ are such that $f_1 + f_2 + \cdots + f_n = 0$, then there exist norm preserving extensions $F_i$ of $f_i$ to $X$ such that $F_1 + F_2 + \cdots + F_n = 0$.*

The following theorem plays key roles in the proofs of main theorems in Section 3. Since Lima's proof of the theorem gives interesting consequences, we duplicate his proof here.

THEOREM 2.3 [10]. *Suppose a closed subspace $M$ of a Banach space $X$ is weakly Hahn-Banach smooth in $X$. If $M$ has the 3.X.I.P., then there is a norm-one projection $P$ in $X^*$ with the kernel $M^\perp$.*

*Proof.* For each $f \in M^*$, let $E(f)$ denote the set of all norm preserving extensions of $f$ to $X$. It suffices to find a linear selection of the map $f \to E(f)$. If $f \in M^*$ attains its norm on $B_M$, let $\hat{f}$ be the unique norm preserving extension of $f$. Then $E(f) = \{\hat{f}\}$. If $f$ and $g$ attain their norms on $B_M$, then by Theorem 2.2 we get $\|f - g\| = \|\hat{f} - \hat{g}\|$. By the Bishop-Phelps theorem [3, P.189] the norm attaining functions in $M^*$ are norm-dense in $M^*$. Hence for each $f \in M^*$, there exists a unique $\tilde{f} \in E(f)$ such that if $f_\alpha \to f$ in norm and each $f_\alpha$ attains its norm, then $\hat{f_\alpha} \to \tilde{f}$ in norm. We can easily see that the selection $f \to \tilde{f}$ is linear. Now define $P : X^* \to X^*$ by $P(f) = (\tilde{f}|_M)$. Then $P$ is a norm-one projection with kernel $M^\perp$.

RÉMARKS. : 1. Observe that if $M$ is Hahn-Banach smooth in the above theorem, then the proof of the theorem becomes much simpler. In this case, for each $f \in M^*$ $E(f) = \{\hat{f}\}$ and $M^\# = \{g \in X^* : \|g\| = \|g|_M\|\}$ is the rang of the projection $P$ constructed above. Hence $M^\#$ is a linear subspace of $X^*$. In fact, we can say more in this connection. Lima [10] proved that for a closed subspace $M$ of a Banach space $X$ $M^\#$ is a linear subspace of $X^*$ if and only if $M$ is Hahn-Banach smooth in $X$ and has the 3.X.I.P.

2. If $X$ is reflexive in the above theorem, by dualizing we can easily see that $M = M^{\perp\perp}$ is the range of the norm one projection $P^*$ on $X$. Thus $M$ itself is nicely complemented in $X$.

THEOREM 2.4 [11]. *For a closed subspace $M$ of a Banach space $X$, the following statements are equivalent.*

(1) *$M^\perp$ is the kernel of a norm-one projection in $X^*$.*
(2) *$M^{\perp\perp}$ is the image of a norm-one projection in $X^{**}$.*
(3) *If $F$ is a finite-dimensional subspace of $X$ and $\varepsilon > 0$, then there exists an operator $T : F \to M$ such that*

$$\text{(i) } Tx = x \qquad \text{for all } x \in F \cap M.$$
$$\text{(ii) } \|T\| \leq 1 + \varepsilon.$$

Let X be a Banach space and let $1 \leq \lambda < \infty$. We say that $X$ has the $\lambda$-approximation property (respectively, $\lambda$-compact approximation property) if, for every compact set $K$ in $X$ and every $\varepsilon > 0$, there exists a finite rank operator (respectively, compact operator) $T$ on $X$ such that $\|T\| \leq \lambda$ and

$$\|Tx - x\| < \varepsilon \qquad \text{for every } x \in K.$$

A Banach space $X$ is said to have the metric approximation property (respectively, metric compact approximation property) if it has the 1-approximation property (respectively, 1-compact approximation property).

The following theorem was proved by J. Johnson [7] under the assumption that $Y$ has the $\lambda$-approximation property. Actually, his proof is valid when $Y$ has the $\lambda$-compact approximation property.

THEOREM 2.5. *Let $X$ and $Y$ be Banach spaces. If $Y$ has the $\lambda$-compact approximation property for some $\lambda \geq 1$, then there is a projection $P$ on $L(X,Y)^*$ such that $\|P\| \leq \lambda$, the range of $P$ is isomorphic to $K(X,Y)^*$ (isometric if $\lambda = 1$) and the kernel of $P$ is $K(X,Y)^\perp$.*

*Proof.* We need only to replace finite rank operators by compact operators in Johnson's proof of Lemma 1 in [7].

## 3. Main results

THEOREM 3.1. *Let $X$ and $Y$ be Banach spaces. If $K(X,Y)$ is weakly Hahn-Banach smooth in $L(X,Y)$ and has the 3.$L(X,Y)$. intersection property, then $B_{K(X,Y)}$ is dense in $B_{L(X,Y)}$ in the topology of uniform convergence on compact sets in $X$.*

*Proof.* Let $P$ be a norm-one projection on $L(X,Y)^*$ with kernel $K(X,Y)^\perp$ as in Theorem 2.3. Then $P^*$ is a norm-one projection on $L(X,Y)^{**}$ with range $K(X,Y)^{\perp\perp}$.

If $T \in L(X,Y) \subseteq L(X,Y)^{**}$ with $\|T\| \leq 1$, then we have $\|P^*T\| \leq 1$ and

$$P^*T \in K(X,Y)^{\perp\perp} = K(X,Y)^{**}.$$

By the Goldstin's theorem, there exists a net $\{T_\alpha\}$ in $K(X,Y)$ such that $\|T_\alpha\| \leq 1$ for all $\alpha$ and

$$T_\alpha \to P^*T$$

in the weak*-topology in $L(X,Y)^{**}$. From the construction of $P$ in Theorem 2.3, we can easily see that if $\phi \in L(X,Y)^*$ attains its norm on $B_{K(X,Y)}$, then $\phi$ is in the range of $P$. If $y^* \in Y^*$ attains its norm on $B_Y$, then for each $x \in X$ the linear functional $\phi = y^* \otimes x$ on $L(X,Y)$, defined by $\phi(S) = y^*(Sx)$ for $S \in L(X,Y)$, attains its norm on $B_{K(X,Y)}$ and is in the range of $P$. Hence we have

$$y^*(T_\alpha x) = \phi(T_\alpha) \to (P^*T)\phi = T(\phi) = y^*(Tx).$$

Since by the Bishop-Phelps theorem [3, P.189] functionals in $Y^*$ which attain their norm on $B_Y$ are dense in $Y^*$, $T_\alpha \to T$ in the weak operator topology and hence in the strong operator topology. Since $\|T_\alpha\| \leq 1$ for all $\alpha$, $T_\alpha \to T$ uniformly on compact sets in $X$.

COROLLARY 3.2. *Let $X$ be a Banach space. If $K(X)$ is weakly Hahn-Banach smooth in $L(X)$ and has the 3.$L(X)$. intersection property, then $X$ has the metric compact approximation property.*

In the proof of Theorem 3.1, we see that if $y^* \in Y^*$ attains its norm on $B_Y$, then for each $x \in X$ the functional $\phi = y^* \otimes x$ on $L(X,Y)$ actually attains its norm at a norm-one, rank-one operator. Hence, replacing $K(X,Y)$ and $B_{K(X,Y)}$ by $\overline{F(X,Y)}$ and $B_{\overline{F(X,Y)}}$, respectively, in Theorem 3.1 and Corollary 3.2 we have the following theorem and corollary.

THEOREM 3.3. *Let $X$ and $Y$ be Banach spaces. If $\overline{F(X,Y)}$ is weakly Hahn-Banach smooth in $L(X,Y)$ and has the 3.$L(X,Y)$. intersection property, then $B_{\overline{F(X,Y)}}$ is dense in $B_{L(X,Y)}$ in the topology of uniform convergence on compact sets in $X$.*

COROLLARY 3.4. *Let $X$ be a Banach space. If $\overline{F(X)}$ is weakly Hahn-Banach smooth in $L(X)$ and has the 3.$L(X)$. intersection property, then $X$ has the metric approximation property.*

THEOREM 3.5. *If $X$ is a Banach space for which $K(X)$ is weakly Hahn-Banach smooth in $L(X)$, then the following statements are equivalent.*

(1) *$X$ has the metric compact approximation property.*
(2) *$K(X)^{\perp}$ is the kernel of a norm-one projection in $L(X)^*$.*
(3) *$K(X)$ has the $n.L(X)$. intersection property for all $n \geq 3$.*

*Proof.* (2) $\Rightarrow$ (3) : Suppose $\{B(T_i, r_i)\}_{i=1}^{n}$ is a family of closed $n$ balls in $K(X)$ such that

$$T \in \bigcap_{i=1}^{n} B(T_i, r_i) \neq \emptyset \qquad \text{in } L(X).$$

Let $F = span\{T_1, T_2, \cdots, T_n, T\}$. By Theorem 2.4 given $\varepsilon > 0$ there exists a linear map $\pi : F \to K(X)$ such that $\|\pi\| \leq 1 + \varepsilon/r$, where $r = \max\{r_1, r_2, \cdots, r_n\}$ and

$$\pi(S) = S$$

for all $S \in F \cap K(X)$. Since $T_i \in K(X)$, $\|T_i - \pi(T)\| \leq r_i + \varepsilon$ for $i = 1, 2, \cdots, n$ and hence

$$\bigcap_{i=1}^{n} B(T_i, r_i + \varepsilon) \neq \emptyset \qquad \text{in } K(X).$$

The implication (3) $\Rightarrow$ (1) follows from Corollary 3.2.

Finally, the implication (1) $\Longrightarrow$ (2) follows from Theorem 2.5.

From the proof of the above theorem and Corollary 3.4, we have the following :

THEOREM 3.6. *Let $X$ be a Banach space for which $\overline{F(X)}$ is weakly Hahn-Banach smooth in $L(X)$. Then the statements below are related as follows : (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5). In addition if $X$ has the approximation property then they are all equivalent.*

(1) *$\overline{F(X)}$ has the 3.$L(X)$. intersection property.*
(2) *$X$ has the metric approximation property.*
(3) *$X$ has the metric compact approximation property.*
(4) *$K(X)^{\perp}$ is the kernel of norm-one projection in $L(X)^*$.*
(5) *$K(X)$ has the n.$L(X)$. intersection property .*

*Proof.* We need to prove only the implication (5) $\Rightarrow$ (1). To do this, it suffices to prove that $\overline{F(X)} = K(X)$. Suppose $X$ has the approximation property, and let $T$ be a compact operator on $X$. Since $\overline{T(B_X)}$ is a compact subset of $X$, for each $\varepsilon > 0$, there exists $S \in F(X)$ such that

$$\|STx - Tx\| < \varepsilon \qquad \text{for all } x \in B_X,$$

*i.e.*, $\|ST - T\| < \varepsilon$. Since $ST \in F(X)$, $T \in \overline{F(X)}$ and it follows that $\overline{F(X)} = K(X)$.

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