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A SIMPLE PROOF OF BÔCHER'S THEOREM

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1. Introduction

A real-valued function u defined on an open subset Ω of $\mathbb{R}^n$ $(n \geq 2)$ is said to be harmonic in Ω if u is twice continuously differentiable and $\Delta u = 0$ in Ω, where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial {x_i}^2}$. Let U denote the open unit ball in $\mathbb{R}^n$ and $E(x)$ denote the fundamental solution of Δ, say,

$$E(x) = \begin{cases} C_1 \log 1/|x| & \text{if } n = 2; \\ C_2 |x|^{2-n} & \text{if } n > 2. \end{cases}$$

Bôcher's theorem characterizes the behavior of the positive harmonic functions in some neighborhood of an isolated singularity as follows:

BÔCHER'S THEOREM. *Let u be positive and harmonic in $U \backslash \{0\}$. Then there exist a harmonic function v in U and a constant $a \geq 0$ such that*

$$u(x) = v(x) + aE(x), \qquad x \in U \backslash \{0\}.$$

In [2] we generalize this theorem for the harmonic function in $U \backslash \{0\}$ without restriction of positivity as follows:

GENERALIZED BÔCHER'S THEOREM ([2]). *Let u be harmonic in the deleted unit ball $U \backslash \{0\}$. Then there exist a function v harmonic in U and constants a_α such that for every $\varepsilon > 0$*

$$u(x) = v(x) + \sum_{\alpha} a_\alpha \partial^\alpha E(x), \qquad x \in U \backslash \{0\}$$

$$|a_\alpha| \leq C_\varepsilon \frac{\varepsilon^{|\alpha|}}{\alpha!}, \qquad \alpha \in \mathbb{N}_0^n.$$

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Also, the expression of u is unique.

Here we use the following multi-index notations: For every multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}_0^n$ where $\mathbb{N}_0$ is the set of nonnegative integers; $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$; $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$; $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}$, with $\partial_j = \frac{\partial}{\partial x_j}$.

In this paper we apply the above generalized Bôcher's theorem to give another simple proof of Bôcher's theorem. For this we made use of the elementary definitions and properties of Sato's hyperfunction in Section 2, which were given in [2, 3 and 4]. In Section 3, we give a different proof of Bôcher's theorem as a main theorem.

2. Preliminaries

We first give brief definitions for the hyperfunctions as in [3] and [4].

DEFINITION 2.1. If $K \subset \mathbb{C}^n$ is a compact set, then $A'(K)$, the space of analytic functionals supported by K, is the space of linear forms u on the space A of entire analytic functions in $\mathbb{C}^n$ such that for every neighborhood ω of K

$$|u(\phi)| \le C_\omega \sup_\omega |\phi|, \qquad \phi \in A.$$

DEFINITION 2.2. Let Ω be a bounded open set in $\mathbb{R}^n$. Then

(1) The space $B(\Omega)$ of hyperfunctions in Ω is defined by

$$B(\Omega) = A'(\overline{\Omega}) / A'(\partial\Omega).$$

(2) For $u \in A'(\overline{\Omega})$ the support of the class $\dot{u}$ of u in $B(\Omega)$ is defined by $\operatorname{supp} \dot{u} = \Omega \cap \operatorname{supp} u$.

Also, we need the following generalized Bôcher's theorem in [2].

THEOREM 2.3. *Let u be harmonic in the deleted unit ball $U \backslash \{0\}$. Then there exist a function v harmonic in U and constants a_α such that for every $\varepsilon > 0$*

$$u(x) = v(x) + \sum_\alpha a_\alpha \partial^\alpha E(x), \qquad x \in U \backslash \{0\}$$

$$|a_\alpha| \le C_\varepsilon \frac{\varepsilon^{|\alpha|}}{\alpha!}, \qquad \alpha \in \mathbb{N}_0^n.$$

Also, the expression of u is unique.

3. Another proof of Bôcher's theorem

We shall now give another proof of Bôcher's theorem by using the result of Theorem 2.3. For this we need the following lemmas.

LEMMA 3.1. *Suppose that u is positive and harmonic in $U\setminus\{0\}$. Then there exist positive constants a and b such that*

$$u(x) \leq aE(x) + b \quad \text{in } 0 < |x| \leq \frac{1}{2}.$$

Proof. Let S denote the unit sphere in $\mathbb{R}^n$ and σ surface area measure. Define the spherical mean of u by

$$M[u](x) = \frac{1}{\sigma(S)} \int_S u(|x|\omega)d\sigma(\omega), \qquad x \in U\setminus\{0\}.$$

Then $M[u]$ is also positive and harmonic in $U\setminus\{0\}$. Furthermore, $M[u]$ is a radial harmonic function in $U\setminus\{0\}$. Hence, $M[u]$ satisfies the differential equation

$$\frac{n-1}{r}f'(r) + f''(r) = 0$$

for $0 < r = |x| < 1$. From this differential equation we see that for some constants a_0 and b_0

$$M[u](x) = a_0 E(x) + b_0.$$

Harnack's inequality (see [6, Theorem 10.3]) implies that there exists a constant $\alpha > 0$ such that for every $0 < r < 1$

$$\alpha u_r(x) < u_r(y), \qquad |x| = |y| = \frac{1}{2}$$

where $u_r(x) = u(rx)$. Then it follows that

$$\alpha u(x) < u(y), \qquad 0 < |x| = |y| \leq \frac{1}{2}.$$

Let ω_0 be the point on S such that $\min_{\omega \in S} u(|x|\omega) = u(|x|\omega_0)$. Then, for $0 < |x| \le 1/2$

$$M[u](x) \ge u(|x|\omega_0) > \alpha u(x).$$

Therefore, we can choose constants a and b such that

$$u(x) \le aE(x) + b, \qquad 0 < |x| \le \frac{1}{2}.$$

Since u is positive the positivities of a and b are necessary, which is required.

We can easily expect that the property of $E(x)$ in the following lemma holds. But its proof may not be so simple if we do not use generalized functions. Here we prove this lemma by using the elementary idea of the generalized functions.

LEMMA 3.2. *Let the harmonic function* $J(x) = \sum_{\alpha} a_\alpha \partial^\alpha E(x)$ *be bounded below in* $\Omega = \{x \mid 0 < |x| \le r\}$ *and* $0 < r < 1$. *Then* $a_\alpha = 0$ *for all* $|\alpha| \ne 0$.

Proof. Since $J(x) + a > 0$ for some constant a it follows from Lemma 3.1 that

$$|J_0(x)| \le AE(x) + B, \qquad x \in \Omega$$

for some$A, B > 0$, where $J_0(x) = \sum_{|\alpha| \ne 0} a_\alpha \partial^\alpha E(x)$ This means that $J_0(x)$ is an integrable function in the ball $B_r = \{x \mid |x| \le r\}$, since $E(x)$ is integrable in B_r. Define a functional on the space $C_0^\infty(B_r)$ of infinitely differentiable functions with support in B_r by

$$J_1(\phi) = \int_{|x| \le r} J_0(x) \Delta \phi(x) dx, \qquad \phi \in C_0^\infty(B_r).$$

Then if we take C_1 to be $L^1(B_r)$-norm of $J_0(x)$ then

$$|J_1(\phi)| \le C_1 \sup_{|x| \le r} |\Delta \phi(x)|, \qquad \phi \in C_0^\infty(B_r).$$

But, in a different point of view, using the integration by parts we have

$$J_1(\phi) = \int_{|x|\le r} \Delta J_1(x)\phi(x)dx$$
$$= \sum_{|\alpha|\ne 0} a_\alpha \partial^\alpha \phi(0), \qquad \phi \in C_0^\infty(B_r).$$

Borel theorem ([5, Theorem 38.1]) states that for any sequence (a_α) there exists a C^∞ function $\psi(x)$ such that $\partial^\alpha\psi(0) = \overline{a}_\alpha$, for all $\alpha \in \mathbb{N}_0^n$. In fact, we may assume that $\psi \in C_0^\infty(B_r)$ by multiplication with a cutoff function. Putting $\phi_n(x) = \psi(nx)$ we obtain that

$$|J_1(\phi_n)| \le C_1 n^2 \sup_{|x|\le r} |\Delta\psi(x)|$$

and

$$J_1(\phi_n) = \sum_{|\alpha|\ne 0} a_\alpha \partial^\alpha \phi_n(0) = \sum_{|\alpha|\ne 0} |a_\alpha|^2 n^{|\alpha|}$$

Then from the relation

$$|a_\alpha|^2 n^{|\alpha|} \le C_1 n^2 \sup_{|x|\le r} |\Delta\psi(x)|,$$

for all $n \in \mathbb{N}_0$ we see that $a_\alpha = 0$ for $|\alpha| > 2$. Thus $J_1(x)$ is of the form

$$J_1(x) = \sum_{i=1}^{n} b_i \frac{\partial E(x)}{\partial x_j} + \sum_{i,j=1}^{n} c_{ij} \frac{\partial^2 E(x)}{\partial x_i \partial x_j}.$$

Since the absolute value of each derivative of $E(x)$ goes to infinity as $x \to 0$ more rapidly than $E(x)$, the inequality holds only if $b_i = 0$ and $c_{ij} = 0$ for all i, j, which is required.

Finally we give a different proof of Bôcher's theorem.

THEOREM 3.3. *Let u be positive and harmonic in $U\setminus\{0\}$. Then there exist a harmonic function v in U and a constant $a \ge 0$ such that*

$$u(x) = v(x) + aE(x), \qquad x \in U\setminus\{0\}.$$

Proof. By Theorem 2.3 there exist a harmonic function v in U and constants a_α such that for every $\varepsilon > 0$

$$u(x) = v(x) + \sum_\alpha a_\alpha \partial^\alpha E(x), \qquad x \in U\backslash\{0\}$$

$$|a_\alpha| \le C_\varepsilon \varepsilon^{|\alpha|}/\alpha!, \qquad \alpha \in \mathbb{N}_0^n.$$

Putting $J(x) = \sum_\alpha a_\alpha \partial^\alpha E(x)$, we can easily see that $J(x)$ is harmonic in $U\backslash\{0\}$. Since v is continuous in U, $J(x)$ is bounded below in $B_{\frac{1}{2}}\backslash\{0\}$. Then Lemma 3.2 implies that $J(x) = a_0 E(x)$. The constant a_0 must be nonnegative. For, otherwise, $u(x) \to -\infty$ as $x \to 0$, which would violate the positivity of u. Thus, the proof is complete.

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