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POLYNOMIAL AND RATIONAL G ISOMORPHISMS OF REAL ALGEBRAIC G VECTOR BUNDLES AND REAL ALGEBRAIC G VARIETIES

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1. Introduction

Let G be a real algebraic group. In this paper we consider two isomorphisms of real algebraic G varieties and real algebraic G vector bundles.

Let X and Y be algebraic G varieties. We say that X is *isomorphic* to Y with respect to a *polynomial* G (resp. *rational* G) *variety isomorphism* if there exist polynomial G (resp. rational G) maps $f : X \longrightarrow Y, h : Y \longrightarrow X$ such that $f \circ h = id, h \circ f = id$. Here a rational map means a fraction of polynomial maps with nowhere vanishing denominator.

Let η and ζ be algebraic G vector bundles. We say that η is *isomorphic* to ζ with respect to a *polynomial* G (resp. *rational* G) *vector bundle isomorphism* if there exist polynomial G (resp. rational G) vector bundle maps $f : \eta \longrightarrow \zeta$, $h : \zeta \longrightarrow \eta$ such that $f \circ h = id, h \circ f = id$. In particular, an algebraic G variety (resp. algebraic G vector bundle) is called *rationally G linearizable* (resp. *rationally G trivial*) if it is isomorphic to some algebraic G module (resp. some trivial G vector bundle) with respect to a rational G variety isomorphism (resp. rational G vector bundle isomorphism).

In the complex category, G.W. Schwarz [7] constructed continuous families of algebraic G vector bundles over some G modules, and solved the linearity problem negatively. M. Masuda and T. Petrie [2][3] introduced an invariant of a polynomial G vector bundle isomorphism, and constructed another such families using this invariant. It is known in [4] that there exists a non-linearizable action of $D_n \times \mathbf{Z}_2$ ($n \geq 6$) on

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$\mathbf{R}^4$, and that there exists a continuous family of inequivalent actions of $D_n \times \mathbf{Z}_2$ ($n \geq 8$) on $\mathbf{R}^4$. Here D_n means the dihedral group of order $2n$ and $\mathbf{Z}_2$ denotes the cyclic group of order two.

The following proves that we can drop $\mathbf{Z}_2$ factor.

THEOREM 1.1. (1) *Let n be even. If $n \geq 10$ then there exists a non-linearizable action of D_n on $\mathbf{R}^4$. If $n \geq 18$ then there exists a continuous family of inequivalent D_n actions on $\mathbf{R}^4$.*
(2) *Any action constructed in (1) is rationally D_n linearizable.*

Theorem 1.1 shows that the above two G variety isomorphisms are quite different. In the complex category, these isomorphisms between algebraic G modules are the same.

Let B, F, S be algebraic G modules. For an algebraic G module M, $\underline{\mathbf{M}}$ denotes the trivial G vector bundle over B with fiber M. Let $Vec(B, F; S)$ be the set of all algebraic G vector bundles η on B with zero fiber F such that $\eta \oplus \underline{\mathbf{S}} \cong \underline{\mathbf{F}} \oplus \underline{\mathbf{S}}$, where $\cong$ means a polynomial G vector bundle isomorphism. Let $VEC(B, F; S)$ denote the set of all polynomial G vector bundle isomorphism classes of $Vec(B, F; S)$.

It is known in [6][8] that, if we forget the action, every element of $Vec(B, F; S)$ is isomorphic to the trivial vector bundle $\underline{\mathbf{F}}$ with respect to a polynomial vector bundle isomorphism.

Let G be the semidirect product of $(\mathbf{R}^*)^q$ and the symmetric group S_q of q letters, where $q \geq 2$. For $m \in \mathbf{Z}$, the q-dimensional real algebraic G module V_m is defined as follows:

$$(g_1, \ldots, g_q)(x_1, \ldots, x_q) = (g_1^m x_1, \ldots, g_q^m x_q)$$

for any $(g_1, \ldots, g_q) \in (\mathbf{R}^*)^q, (x_1, \ldots, x_q) \in \mathbf{R}^q$, and S_q acts by permutating coordinates.

THEOREM 1.2. (1) *$VEC(V_1 \oplus V_{-1}, V_m \oplus V_{-m}; \underline{\mathbf{R}})$ contains a continuous family of dimension $m-1$ if $m \geq 2$. Furthermore, there exists a non-linearizable actions of $G \times \mathbf{Z}_2$ on $\mathbf{R}^{4q}$.*
(2) *Any member of the family constructed in (1) is rationally G trivial.*
(3) *The action on the total space of any element of the family is rationally G linearizable.*

Theorem 1.2 shows that the above two G vector bundle isomorphisms are quite different. In the complex category, if the base space is a complex algebraic G module then these isomorphisms are the same.

We get the following when G is compact.

THEOREM 1.3. *Let G be a compact real algebraic group and let η be a real algebraic G vector bundle over a real algebraic G module B. If there exists some one-dimensional G module S such that $\eta \oplus \mathbf{S}$ is rationally G trivial, then η is rationally G trivial.*

The present paper is organized as follows. We introduce a *realification* of a complex algebraic G vector bundle in section 2. In section 3, we construct a non-trivial family of $VEC(B,F;S)$. In section 4, we prove that each element of such family is rationally G trivial, and Theorem 1.3.

2. Complexifications and realifications

Recall the complexification of real algebraic G varieties and real algebraic G vector bundles [4].

DEFINITION 2.1. Let $X \subset \mathbf{R}^n$ be a real algebraic variety with the coordinate ring $O(X)$. The complex variety $X_C \subset \mathbf{C}^n$ is called the *complexification* of X if it consists of the common zeros of all elements of $O(X)$ regarded as the map from $\mathbf{C}^n$ to $\mathbf{C}$.

As easily checked, G_C is a (resp. reductive) complex algebraic group if G is a (resp. compact) real algebraic group, and X_C is a G_C variety if X is a G variety.

DEFINITION 2.2. (1) Let G be an algebraic group and let B, F, S be algebraic G modules. We define $sur(\underline{\mathbf{F}} \oplus \underline{\mathbf{S}}, \underline{\mathbf{S}})$ by the set of all polynomial G vector bundle maps $L : \underline{\mathbf{F}} \oplus \underline{\mathbf{S}} \longrightarrow \underline{\mathbf{S}}$ with polynomial G splitting.
(2) Let G be a real algebraic group and let B, F, S be real algebraic G modules. For $L \in sur(\underline{\mathbf{F}} \oplus \underline{\mathbf{S}}, \underline{\mathbf{S}})$, the natural extension $L_C : \underline{\mathbf{F}}_C \oplus \underline{\mathbf{S}}_C \longrightarrow \underline{\mathbf{S}}_C$ is in $sur(\underline{\mathbf{F}}_C \oplus \underline{\mathbf{S}}_C, \underline{\mathbf{S}}_C)$. For $E = kerL \in Vec(B,F;S)$, we define $E_C = kerL_C \in Vec(B_C, F_C; S_C)$ and call it the *complexification* of E.

In the complex category, it is known in [1] that any surjective polynomial G vector bundle map $\underline{\mathbf{F}} \oplus \underline{\mathbf{S}} \longrightarrow \underline{\mathbf{S}}$ admits a polynomial G splitting when G is reductive.

DEFINITION 2.3. Let G be a real algebraic group and let B, F, S be complex G_C modules. Suppose $\phi \in sur(\underline{\mathbf{F}} \oplus \underline{\mathbf{S}}, \underline{\mathbf{S}})$. Then $\eta = ker\phi$ is a complex algebraic G_C vector bundle over B. A real algebraic G vector bundle η' over a real algebraic G module $B' \subset B$ with $\eta' \subset \eta$ is called a *realification* of η if the inclusion $\eta' \longrightarrow \eta$ extends a polynomial G_C vector bundle isomorphism $(\eta')_C \longrightarrow \eta$.

This realification is not uniquely defined and does not always exist. The next proposition gives a sufficient condition that a realification exists.

PROPOSITION 2.4. *Let G be a compact real algebraic group and let B, F, S be complex algebraic G_C modules. Suppose that $\eta = ker\phi$ with $\phi \in sur(\underline{\mathbf{F}} \oplus \underline{\mathbf{S}}, \underline{\mathbf{S}})$. If there exist involutive antiholomorphic automorphisms $\tau_B : B \longrightarrow B, \tau_F : F \longrightarrow F, \tau_S : S \longrightarrow S$ such that*

(1) *the fixed point sets of them are not empty*
(2) *they commute ϕ and the G_C action*

then $\eta' = \eta \cap (B' \times (F' \times S'))$ is a realification of η, where B', F', S' are the fixed point sets of τ_B, τ_F, τ_S, respectively.

Proof. By Theorem 2.3.7.6 [5], B', F', S' are real forms of B, F, S, respectively. Hence they are real algebraic G modules because of (2). Therefore the inclusion $B' \times (F' \times S') \longrightarrow B \times (F \times S)$ extends a polynomial G_C vector bundle isomorphism $h : (B')_C \times ((F')_C \times (S')_C) \longrightarrow B \times (F \times S)$. Let $\phi' : B' \times (F' \times S') \longrightarrow B' \times S'$ be the restriction of ϕ on $B' \times (F' \times S')$. By $\phi \in sur(\underline{\mathbf{F}} \oplus \underline{\mathbf{S}}, \underline{\mathbf{S}})$ and (2), $\phi' \in sur(\underline{\mathbf{F}'} \oplus \underline{\mathbf{S}'}, \underline{\mathbf{S}'})$. Set $\eta' = ker\phi'$. Then η' is a real algebraic G vector bundle over B', and the inclusion $\eta' \longrightarrow \eta$ extends a polynomial G_C vector bundle isomorphism $h|_{(\eta')_C} : (\eta')_C \longrightarrow \eta$. $\square$

3. Polynomial G isomorphisms

Recall some notations and results [3]. Suppose that G is an algebraic group, and that B, F, S are algebraic G modules. Let $mor(\underline{\mathbf{F}}, \underline{\mathbf{S}})$ be the set of polynomial G vector bundle maps from $\underline{\mathbf{F}}$ to $\underline{\mathbf{S}}$. Any $L \in sur(\underline{\mathbf{F}} \oplus \underline{\mathbf{S}}, \underline{\mathbf{S}})$ has components $L(F, S)$ and $L(S, S)$.

DEFINITION 3.1. Let $Z \subset mor(\underline{\mathbf{F}}, \underline{\mathbf{S}})$.
(1) $sur_Z(\underline{\mathbf{F}} \oplus \underline{\mathbf{S}}, \underline{\mathbf{S}}) = \{L \in sur(\underline{\mathbf{F}} \oplus \underline{\mathbf{S}}, \underline{\mathbf{S}}) | L(F, S) \in Z\}$.
(2) $VEC_Z(B, F; S) = \{kerL \in VEC(B, F; S) | L \in sur_Z(\underline{\mathbf{F}} \oplus \underline{\mathbf{S}}, \underline{\mathbf{S}})\}$.
(3) For a trivial G vector bundle $\underline{\mathbf{M}}$, $aut(\underline{\mathbf{M}})$ denotes the group of polynomial G vector bundle automorphisms $\underline{\mathbf{M}} \longrightarrow \underline{\mathbf{M}}$.
(4) For $\phi \in mor(\underline{\mathbf{F}}, \underline{\mathbf{S}})$,

$$R(\phi)_* = \{T \in end(\underline{\mathbf{S}}) | \phi + T \in sur(\underline{\mathbf{F}} \oplus \underline{\mathbf{S}}, \underline{\mathbf{S}}), T(0) = id\}.$$

Let $G = D_n = \mathbf{Z}_n \rtimes \mathbf{Z}_2 \subset \mathbf{C}^* \rtimes \mathbf{Z}_2$, and let g be a generator of $\mathbf{Z}_n$, h a generator of $\mathbf{Z}_2$. For $m \in \mathbf{Z}$, the two-dimensional complex algebraic G module U_m defined by

$$g(a, b) = (\zeta^m a, \zeta^{-m} b), h(a, b) = (b, a),$$

where $(a, b) \in U_m$ $(= \mathbf{C}^2)$ and $\zeta = \exp(2\pi\sqrt{-1}/n)$.

We are concerned with the case when $B = U_2, F = U_m$ $(0 \leq 2m \leq n, m$ is odd $)$, $S = U_1$. Let $\phi \in mor(\underline{\mathbf{F}}, \underline{\mathbf{S}})$ be $\phi(a, b)(x, y) = (b^k x, a^k y)$, where $k = (m-1)/2$. Put

$$N = \begin{cases} n \text{ if } n \text{ is odd} \\ n/2 \text{ if } n \text{ is even} \end{cases}.$$

One can easily check that any element $T \in end(\underline{\mathbf{S}})$ is uniquely expressed as $T = \Sigma_{i=1}^3 T_i \xi_i$, where $T_i \in O(U_2)^G, \xi_0(a, b)(s, t) = (s, t), \xi_1(a, b)(s, t) = (at, bs), \xi_2(a, b)(s, t) = (b^{N-1}t, a^{N-1}s), \xi_3(a, b)(s, t) = (a^N s, b^N t)$ for any $(a, b) \in B, (s, t) \in S$.

THEOREM 3.2. [3] *Let* $E_\phi(T) = ker(\phi + T)$, $c^* T_i(x) = T_i(cx), \Delta = ab, \Xi = a^N + b^N$, *and*

$$\beta = \begin{cases} [(n-2m)/4] & \textit{if } \; 2m \leq n \leq 4m \textit{ and } n \textit{ is even} \\ (m-1)/2 & \textit{if } \; 2m \leq n \leq 4m \textit{ and } n \textit{ is odd, or } 4m < n \end{cases}.$$

(1) *If* $T_0'T_1 \equiv T_0T_1'$ *mod* (Δ^β, Ξ), *then* $E_\phi(T)$ *is isomorphic to* $E_\phi(T')$ *with respect to a polynomial* G *vector bundle isomorphism.*
(2) *If* n *is even and there is a non-zero constant* $c \in \mathbf{C}$ *such that* $(c^*T_0')T_1 \equiv c(c^*T_1')T_0$ *mod* (Δ^β, Ξ), *then the total space of* $E_\phi(T)$ *is isomorphic to that of* $E_\phi(T')$ *with respect to a polynomial* G *variety isomorphism.* □

COROLLARY 3.3. [3] *The set of actions of* G *induced from elements in*
$VEC_\phi(U_2, U_m; U_1)$ *contains a continuous family of dimension* $\beta - 1$ *if* n *is even.* □

We define the involutive antiholomorphic automorphism of U_m by

$$\tau_m : U_m \longrightarrow U_m, \tau_m(a,b) = (\overline{b}, \overline{a}).$$

Let U_m' be the fixed point set of τ_m. Since they satisfy the assumptions of Proposition 2.4, we have the next result.

THEOREM 3.4. (1) $VEC(U_2', U_m'; U_1')$ *contains a continuous family of dimension* β.
(2) *The set of actions of* G *obtained from elements in* $VEC(U_2', U_m'; U_1')$ *contains a continuous family of dimension* $\beta - 1$ *if* n *is even.* □

Theorem 1.1 (1) follows from Theorem 3.4.

Let G be the semidirect product of $(\mathbf{C}^*)^q$ and S_q, where $q \geq 2$. Let W_m ($m \in \mathbf{Z}$) be the q-dimensional complex algebraic G module defined as follows:

$$(g_1, \ldots, g_q)(x_1, \ldots, x_q) = (g_1^m x_1, \ldots, g_q^m x_q)$$

for any $(g_1, \ldots, g_q) \in (\mathbf{C}^*)^q, (x_1, \ldots, x_q) \in \mathbf{C}^q$, and S_q acts by permutating coordinates.

Let $B = W_1 \times W_{-1}, F = W_m \times W_{-m}, S = \mathbf{C}$, where m is a positive integer. We define $\phi \in mor(\underline{\mathbf{F}}, \underline{\mathbf{S}})$ by

$$\phi(a,b)(x,y) = \Sigma_{i=1}^q (a_i^m y_i + b_i^m x_i), (a,b) \in B, (x,y) \in F.$$

Then $O(B)^G = \mathbf{C}[\sigma_1, \ldots, \sigma_q]$ and $R(\phi)_* = \{T \in O(B)^G | T(0) = 1\}$, where σ_i is i-th elementary symmetric polynomial of $a_1b_1, \ldots, a_qb_q$.

THEOREM 3.5. [3] *$VEC_\phi(B, F; S)$ contains a continuous family of dimension $m-1$.* $\square$

Proof of Theorem 1.2. (1) Let $\tau_m : W_m \longrightarrow W_m$ be $\tau_m(a,b) = (\overline{a}, \overline{b})$. Then it is an involutive antiholomorphic automorphism of W_m, and the fixed point set of it is V_m. Therefore, Theorem 1.2 (1) follows from Theorem 3.5 and Proposition 2.4. $\square$

4. Rational G triviality

The following is an elementary lemma, and we leave the proof to the reader.

LEMMA 4.1. *Let G be a real algebraic group and let B, F, S be real algebraic G modules. Suppose that $\phi \in mor(\underline{\mathbf{F}}, \underline{\mathbf{S}})$ and $T \in end(\underline{\mathbf{S}})$ satisfy $\phi + T \in sur(\underline{\mathbf{F}} \oplus \underline{\mathbf{S}}, \underline{\mathbf{S}})$. If $\det T$ is nowhere vanishing on B then $ker(\phi + T)$ is rationally G trivial.* $\square$

Proof of Theorem 1.1 (2). We describe the family of bundles constructed in Theorem 3.4 (1). Let $B' = U_2', F' = U_m', S' = U_1'$. As easily checked, any element $T' \in end(\underline{\mathbf{S}'})$ is uniquely expressed as $T' = \Sigma_{i=1}^3 T_i' \xi_i'$, where $T_i' \in O(B')^G = \mathbf{R}[a\overline{a}, a^N + \overline{a}^N]$ and $\xi_i' = \xi_i|(B' \times S')$. Recall that $\phi \in mor(\underline{\mathbf{U}_m}, \underline{\mathbf{U}_1})$ is $\phi(a,b)(x,y) = (b^k x, a^k y)$. Let $\phi' \in mor(\underline{\mathbf{F}'}, \underline{\mathbf{S}'})$ be the restriction of ϕ on $\underline{\mathbf{F}'}$. For $T" \in R(\phi')_*$, the real algebraic G vector bundle $E_{T"} := ker(\phi' + T")$ is described by

$$E_{T"} = \{(a, \overline{a}, f, \overline{f}, s, \overline{s}) \in B' \times F' \times S' |\, a^k \overline{f} + T_0' \overline{s} + T_1' \overline{a} s + T_2' a^{N-1} s + T_3' \overline{a}^N \overline{s} = 0\}.$$

Define $A \in aut(\underline{\mathbf{F}'} \oplus \underline{\mathbf{S}'})$ by

$$\begin{aligned} &A(a, \overline{a}, f, \overline{f}, s, \overline{s}) \\ &= (a, \overline{a}, f - a^{N-k-1} T_2' s - a^{N-k} T_3' \overline{s}, \overline{f} - \overline{a}^{N-k-1} T_2' \overline{s} - \overline{a}^{N-k} T_3' s, s, \overline{s}). \end{aligned}$$

Then the image $E'_{T"}$ of $E_{T"}$ is

$$\{(a, \overline{a}, f, \overline{f}, s, \overline{s}) \in B' \times F' \times S' |\, a^k \overline{f} + T_0' \overline{s} + T_1' \overline{a} s = 0\}.$$

For a positive real number u and a positive integer l greater than k, we consider the next transformation.

$$\begin{cases} f = f + ua^l\overline{a}^{l-k}T_0's \\ \overline{f} = \overline{f} + u\overline{a}^l a^{l-k}T_0'\overline{s} \end{cases}$$

This induces $A' \in aut(\underline{\mathbf{F}'} \oplus \underline{\mathbf{S}'})$, and the image $E"_{T"}$ of $E'_{T"}$ is

$$\{(a,\overline{a},f,\overline{f},s,\overline{s}) \in B' \times F' \times S' | a^k\overline{f} + (1+ut^l)T_0'\overline{s} + T_1'\overline{a}s = 0\},$$

where $t = a\overline{a}$.

Since $a^k\overline{f}+(1+ut^l)T_0'\overline{s}+T_1'\overline{a}s = 0$ implies $\overline{a}^k f+(1+ut^l)T_0's+T_1'a\overline{s} = 0$, the matrix representation and the determinant are the following:

$$\begin{pmatrix} T_1'\overline{a} & (1+ut^l)T_0' \\ (1+ut^l)T_0' & T_1'a \end{pmatrix} \begin{pmatrix} s \\ \overline{s} \end{pmatrix} = \begin{pmatrix} -a^k\overline{f} \\ -\overline{a}^k f \end{pmatrix},$$

$$\det \begin{pmatrix} T_1'\overline{a} & (1+ut^l)T_0' \\ (1+ut^l)T_0' & T_1'a \end{pmatrix} = t(T_1')^2 - (1+ut^l)^2(T_0')^2.$$

Since $T_0'(0) = 1$, det < 0 for any $(a,\overline{a}) \in B'$ if l and u are sufficiently large. By Lemma 4.1, each element in the family constructed in Theorem 3.4 (1) is rationally G trivial. Therefore Theorem 1.1 (2) is proved. □

Proof Theorem 1.2 (2) and (3). Let $B = V_1 \times V_{-1}, F = V_m \times V_{-m}, S = \mathbf{R}$. Let $\phi'(a,b)(x,y) = \Sigma_{i=1}^q(a_i^m y_i + b_i^m x_i)$, where $(a,b) \in B, (x,y) \in F$. The following is the explicit description of an element E_f in the set $VEC_{\phi'}(B,F;S)$ constructed in Theorem 1.2 (1):

$$E_f = \{(a,b,x,y,z) \in B \times F \times S | \phi'(a,b)(x,y) + f(\sigma_1,\dots,\sigma_q)z = 0\},$$

where σ_i denotes the i-th elementary symmetric polynomial of $a_1b_1,\dots$ a_qb_q, $f \in \mathbf{R}[\sigma_1,\dots,\sigma_q]$ and $f(0) = 1$.

For a positive real number c and a positive integer l greater than $m/2$,

let

$$y_i' = y_i + ca_i^{2l-m}b_i^{2l}z, 1 \le i \le q.$$

This induces $A \in aut(\underline{\mathbf{F}} \oplus \underline{\mathbf{S}})$, and the image E'_f of E_f is

$$E'_f = \{(a,b,x,y,z) | \phi'(a,b)(x,y) + (c\Sigma_{i=1}^q a_i^{2l} b_i^{2l} + f)z = 0\}.$$

Since $f(0) = 1$, $c\Sigma_{i=1}^q a_i^{2l} b_i^{2l} + f \neq 0$ when c and l are sufficiently large. By Lemma 4.1, E'_f is rationally G trivial, and (3) follows from (2). □

Proof of Theorem 1.3. If dim $B = 0$ then there is nothing to prove. We assume dim $B \geq 1$. Let F denote the zero fiber of η. Since $\eta \oplus \underline{\mathbf{S}}$ is rationally G trivial, there exists a rational G vector bundle isomorphism $j : \eta \oplus \underline{\mathbf{S}} \longrightarrow \underline{\mathbf{F}} \oplus \underline{\mathbf{S}}$. Then we have the following exact sequence:

$$0 \longrightarrow \eta \xrightarrow{i} \underline{\mathbf{F}} \oplus \underline{\mathbf{S}} \xrightarrow{L} \underline{\mathbf{S}} \longrightarrow 0,$$

where i is the composition of the inclusion $\eta \longrightarrow \eta \oplus \underline{\mathbf{S}}$ with j, and that L is the composition j^{-1} with the natural projection $\eta \oplus \underline{\mathbf{S}} \longrightarrow \underline{\mathbf{S}}$. Hence L is a surjective G vector bundle map, and η is isomorphic to ker L with respect to a rational G vector bundle isomorphism. We identify η with ker L.

For any two real algebraic G modules U and V, we denote $mor'(\underline{\mathbf{U}}, \underline{\mathbf{V}})$ by the set of all rational G vector bundle maps from $\underline{\mathbf{U}}$ to $\underline{\mathbf{V}}$. Then, there exists a fundamental isomorphism

$$mor'(\underline{\mathbf{U}}, \underline{\mathbf{V}}) \cong Mor'(B, Hom(U,V))^G,$$

where the right-hand side in this isomorphism is the set of all rational G maps from B to $Hom(U,V)$ with the conjugate action of G.

Since

$$\begin{aligned} mor'(\underline{\mathbf{F}} \oplus \underline{\mathbf{S}}, \underline{\mathbf{S}}) &= mor'(\underline{\mathbf{F}}, \underline{\mathbf{S}}) \oplus mor'(\underline{\mathbf{S}}, \underline{\mathbf{S}}), \\ &\cong Mor'(B, Hom(F,S))^G \oplus Mor'(B, Hom(S,S))^G, \end{aligned}$$

$L : \underline{\mathbf{F}} \oplus \underline{\mathbf{S}} \longrightarrow \underline{\mathbf{S}}$ can be described by $L(b,f,s) = (b, \phi(b)f + T(b)s)$, where $b \in B, f \in F, s \in S, \phi \in Mor'(B, Hom(F,S))^G$ and $T \in Mor'(B, Hom(S,S))^G$. Hence we can write

$$\eta = \{(b,f,s) \in B \times F \times S \mid \phi(b)f + T(b)s = 0\}.$$

Taking the dual spaces of F and S, we have the following correspondence:

$$Mor'(B, Hom(F,S))^G \longrightarrow Mor'(B, Hom(S^*, F^*))^G.$$

Since G is compact, $Mor'(B, Hom(S^*, F^*))^G \cong Mor'(B, Hom(S, F))^G$. Therefore ϕ induces the rational G vector bundle map $\phi^* : \underline{\mathbf{S}} \longrightarrow \underline{\mathbf{F}}$ ($\in mor'(\underline{\mathbf{S}}, \underline{\mathbf{F}}) \cong Mor'(B, Hom(S,F))^G$).

Assume that both $\phi(b)\phi^*(b)$ and $T(b)$ are zero maps on some fiber over $b \in B$. By the construction of ϕ^* and since S is one-dimensional, $\phi(b)\phi^*(b)$ is a zero map if and only if $\phi(b)$ is a zero map. Hence L is not surjective on the fiber over $b \in B$. This shows at least one of them are not zero maps on any fiber.

Since dim $B \geq 1$ and G is compact, there exists a non-trivial G invariant polynomial function on B. Thus for any $n \in \mathbf{N}$ there exists a G invariant polynomial function f on B so that deg $f \geq n$ and that $f(x) > 0$ for any $x \in B$.

Since for any $b \in B$ $\phi(b)\phi^*(b)$ and $T(b)$ are endomorphisms of a one-dimensional real vector space, we regard them as real numbers. Because for any $b \in B$ $\phi(b)\phi^*(b)$ is non-negative and the above two arguments, one can find a G invariant polynomial function l on B such that

$$l(b) > 0 \text{ and } l(b)\phi(b)\phi^*(b) > -T(b) \text{ for any } b \in B.$$

We define the polynomial G vector bundle automorphism A of $\underline{\mathbf{F}} \oplus \underline{\mathbf{S}}$ by

$$A(b, f, s) = (b, f + l(b)\phi^*(b)(s), s).$$

The image η' of η is

$$\{(b, f, s) \in B \times F \times S | \phi(b)f + (l(b)\phi(b)\phi^*(b) + T(b))s = 0\}.$$

Hence it suffices to prove that η' is rationally G trivial. By the choice of l,

$$l(b)\phi(b)\phi^*(b) + T(b) \neq 0 \text{ for any } b \in B.$$

Therefore, by Lemma 4.1, the proof is complete. □

REMARK 4.2. Each member of the family of real algebraic $O_2(\mathbf{R})$ vector bundles constructed in Theorem 2.1 [4] is rationally $O_2(\mathbf{R})$ trivial, and each element of the family of inequivalent real algebraic $O_2(\mathbf{R}) \times \mathbf{Z}_2$ actions on $\mathbf{R}^4$ obtained from Theorem 2.5 [4] is rationally $O_2(\mathbf{R}) \times \mathbf{Z}_2$ linearizable.

It is reasonable that we conjecture the following.

CONJECTURE. *Let G be a compact real algebraic group, and let B, F, S be real algebraic G modules. Every element in $VEC(B, F; S)$ is rationally G trivial.*

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