PICARD GROUP OF A SURFACE IN $\mathbb{P}^3$

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Abstract. We give the optimal lower bound for the Picard number of certain surfaces in the Noether-Lefschetz locus.

1. Introduction

We work over the complex numbers $\mathbb{C}$. A surface or a curve is a projective variety of dimension two or one, respectively. Lefschetz (1,1) Theorem asserts that there is a one to one correspondence between the curves on a smooth surface and the integral (1,1) cohomology classes of the surface. So the Picard group $Pic(S)$ of a smooth surface $S$ in $\mathbb{P}^n$ is identified with $H^{1,1}(S) \cap H^2(S, \mathbb{Z})$, where, of course, $H^2(S, \mathbb{Z})$ is its image under the natural inclusion in $H^2(S, \mathbb{R})$. Moreover, a curve on $S$ which is not a complete intersection of $S$ with another hypersurface in $\mathbb{P}^n$ corresponds to a primitive integral (1,1)-class of $S$. That is, such a curve provides a generator other than the hyperplane class of $S$ in $Pic(S)$.

Noether-Lefschetz theorem([5]) says that $Pic(S) \cong \mathbb{Z}$ for a general surface of degree $d \geq 4$ in $\mathbb{P}^3$.

The word "general" is used in the sense that a property is said to hold at a general point of a projective variety $V$ if the property holds at all the points of $V$ but the points in a countable union of subvarieties of $V$.

Since Lefschetz gave the complete proof of the above theorem, many different ways of proving the same theorem and improvements of the theorem in various directions have been done by several mathematicians. One direction is to study and compute the Picard number of a surface.

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which is the rank of the Picard group. For example, A. Lopez([6]) characterized the Picard group of a general surface in $\mathbb{P}^3$ containing a fixed curve.

In this paper, we will provide a lower bound for the Picard number of certain surfaces(Theorem 3.1). The method is infinitesimal Hodge theoretic and we will use one of M. Green's theorem.

2. Preliminaries

Let $Y_d$ be the set of smooth surfaces of degree $d$ in $\mathbb{P}^3$.

**Definition 2.1.** The *Noether-Lefschetz locus* $\Sigma$ in $Y_d$ is defined as

$$\Sigma = \{ S \in Y_d \mid \text{Pic}(S) \not\cong \mathbb{Z} \}.$$  

M. Green([2]) proved that the codimension of an irreducible component of $\Sigma$ in $Y_d$ is at least $d - 3$ for $d \geq 4$. The following theorem is also proven by M. Green.

**Theorem 2.2 (M. Green).** Let $W \subseteq H^0(\mathcal{O}_{\mathbb{P}^n}(d))$ be a base point free linear subspace of codimension $c$. Let $\mu_k$ denote the multiplication map

$$W \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(k))^\mu_k H^0(\mathcal{O}_{\mathbb{P}^n}(d + \kappa))$$

and

$$c_k = \text{codim} \,(\text{image} \, \mu_k).$$

If $c \leq d$ and $c \kappa_{c-1} \neq 0$, then $c_k = c - k$ for $0 \leq k \leq c$.

3. Main Results

**Theorem 3.1.** Let $d \geq 5$. Let $Z$ be an irreducible component of $\Sigma$, and $c = \text{Codim}_{Y_d} Z$. If $c \leq d$ and $c \kappa_{c-1} \neq 0$, then the Picard number of $S$ is $\geq c - d + 5$, for a smooth point $S$ of $Z$. 
PROOF. Let $T_1$ be the Zariski tangent space of $Z$ at $S$. Since $S$ is in $\Sigma$, there exists a curve $C$ which is not a complete intersection. Let $L = O_S(C)$ and $\gamma = c_1(L) \in H^{1,1}_\text{prim}(S)$. An extension $M$ of the tangent sheaf $\Theta_S$ of $S$ by $O_S$ is defined by the exact sequence

$$0 \to O_S \to M_S \to \Theta_S \to 0$$

with the extension class $c_1(L)$. We can choose $C$ so that the image $T_\gamma$ of $T_1$ under the Kodaira-Spencer map is in the kernel of the map

$$H^1(S, \Theta_S) \to H^2(S, O_S)$$

which is the cup product map with $\gamma = c_1(L)$. Equivalently, the image of $T_\gamma \otimes H^0(S, K_S)$ is contained in $c_1(L) \perp$. Using the standard identification (cf. [1] or [4]), this is the multiplication map

$$\frac{H^0(\mathbb{P}^3, O_{\mathbb{P}^3}(d))}{J_d} \otimes H^0(\mathbb{P}^3, O_{\mathbb{P}^3}(d - 4)) \to \frac{H^0(\mathbb{P}^3, O_{\mathbb{P}^3}(2d - 4))}{J_{2d-4}}$$

where $J_k$ denotes the Jacobian ideal of $S$ in degree $k$. Let $W$ be the preimage of $T_\gamma$ in $H^0(\mathbb{P}^3, O_{\mathbb{P}^3}(d))$. Then the multiplication map

$$W \otimes H^0(\mathbb{P}^3, O_{\mathbb{P}^3}(d - 4)) \to H^0(\mathbb{P}^3, O_{\mathbb{P}^3}(2d - 4))$$

is not surjective. $W$ contains $J_d$ and $J_d$ is base point free since $S$ is smooth. Hence $W$ is a base point free linear subspace of codimension $c$. For $c$ such that $c \leq d$ and $c_{c-1} \neq 0$, the codimension of the image of $W \otimes H^0(\mathbb{P}^3, O_{\mathbb{P}^3}(d - 4))$ is $c - d + 4$ by Theorem (2.3). Hence the codimension of $T_\gamma \otimes H^0(S, K_S)$ in $H^{1,1}_\text{prim}(S)$ is $c - d + 4$. So the $\text{dim} H^{1,1}_\text{prim}(S) \geq c - d + 4$ and rank $\text{Pic}(S) \geq c - d + 5$. \(\square\)

REMARK. It is well known that the component of $\Sigma$ whose generic member $S$ contains a line has codimension $d - 3([3])$. In this case, the inequality in Theorem (3.1) says that rank $\text{Pic}(S) \geq 2$, which is trivial. A. Lopez proved that the equality holds in this case.
Theorem 3.2 (A. Lopez). Let \( d \geq 5 \). Let \( Z_1 \) denote an irreducible component of \( \Sigma \) whose generic member contains a line. Then for a general \( S \) in \( Z_1 \), the Picard number is 2.

His proof can be divided into two steps. The first step is to show that the codimension of \( T_\gamma \otimes H^0(S, K_S) \) in \( H^{1,1}_{prim}(S) \) is 1. The second step shows that this fact implies the Picard number of \( S \) to be 2. By using Theorem (3.1), we can prove the first part as follows.

Lemma 3.3. The codimension of \( T_\gamma \otimes H^0(S, K_S) \) in \( H^{1,1}_{prim}(S) \) is 1.

Proof. Let \( S \) be a smooth point of \( Z_1 \). Then the codimension of \( Z_1 \) is \( d - 3 \). Let \( C \) denote a line contained is \( S \). Then we can choose \( C \) so that the argument in the proof of Theorem (3.1) can hold. Hence for \( \gamma = c_1(O_S(C)) \in H^{1,1}_{prim}(S) \), the codimension of \( T_\gamma \otimes H^0(S, K_S) \) in \( H^{1,1}_{prim}(S) \) is 1. □

For the second step showing how the above lemma implies this theorem, we will restate Lopez’s proof.

Proof. ([7]) By the semicontinuity theorem, it is enough to prove that for each class \( \gamma' \in H^{1,1}_{prim}(S, C) \setminus C_\gamma \), there exists a deformation \( \eta \in T_\gamma \) such that, when we deform \( S \) in the direction of \( \eta \) to a surface \( S' \), the class \( \gamma' \) is not of type \((1, 1)\) in \( H^2(S', \mathbb{Z}) \cong H^2(S, \mathbb{Z}) \).

But, by the Lemma (3.3),

\[
H^{1,1}_{prim}(S, C) \cong C(\gamma) \oplus \text{image}(T_\gamma \otimes H^0(S, K_S)).
\]

Let \( \gamma'' \) be the nonzero component of \( \gamma' \) in \( \text{im}(T_\gamma \otimes H^0(S, K_S)) \). If \( T_\gamma \subset T_{\gamma''} \), then

\[
\gamma'' \in \text{im}(T_\gamma \otimes H^0(S, K_S)) \subset \text{im}(T_{\gamma''} \otimes H^0(S, K_S)) \subset (\gamma'')^\perp
\]

which is a contradiction. □

References


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