A NOTE ON $I$-IDEALS IN $BCI$-SEMIGROUPS

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Abstract. In this paper, we describe the ideal generated by non-empty stable set in a $BCI$-group as a simple form, and obtain an equivalent condition of prime $I$-ideal.

1. Introduction

The notion of $BCK$-algebras was proposed by Y. Imai and K. Iséki in 1966. In the same year, K. Iséki ([3]) introduced the notion of $BCI$-algebra which is a generalization of a $BCK$-algebra. The ideal theory plays an important role in studying $BCK$-algebras and $BCI$-algebras, and some interesting results have been obtained by several authors ([1,2,9]). In particular, the study of prime ideals is also an important part of the theory of $BCK$-algebras ([1]). In 1993, Y.B. Jun and et. al. ([6]) introduced the notion of $BCI$-semigroups/monoid, and studied their properties. They also considered the concept of $I$-ideals and of zero-divisors in $BCI$-semigroups. Some authors ([7, 8]) studied $BCI$-semigroups with the notion of fuzzy (commutative) $I$-ideals. Every $p$-semisimple $BCI$-algebra gives naturally an abelian group by defining $x + y := x \ast (0 \ast y)$, and hence $p$-semisimple $BCI$-semigroup leads to the ring structure. On the while, every ring gives a $BCI$-algebra by defining $x \ast y := x - y$ and hence we can construct a $BCI$-semigroup. Hence the $BCI$-semigroup is a generalization of the ring. In this paper, we describe the ideal generated by a non-empty stable set in a $BCI$-group as a simple form, and obtain an equivalent condition of a prime $I$-ideal. Let us recall definitions and some propertites.

Received March 18, 1996. Revised July 26, 1996.
1991 AMS Subject Classification: 06F35, 06A06.
Key words and phrases: $BCI$-semigroup(group), (prime) $I$-ideal, stable.
* This paper was supported (in part) by BSRI program, MOE, 1995, Project No. BSRI-95-1423.
Definition 1.1 ([6]). A BCI-semigroup is a non-empty set $X$ with two binary operations "$\ast$" and "$\cdot$" and constant 0 satisfying the following axioms:

1. $(X; \ast, 0)$ is a BCI-algebra,
2. $(X, \cdot)$ is a semigroup,
3. the operation "$\cdot$" is distributive (on both sides) over the operation "$\ast$", that is, $x \cdot (y \ast z) = (x \cdot y) \ast (x \cdot z)$ and $(x \ast y) \cdot z = (x \cdot z) \ast (y \cdot z)$ for all $x, y, z \in X$.

Example 1.2. Define two binary operations "$\ast$" and "$\cdot$" on a set $X := \{0, 1, 2, 3\}$ as follows:

\[
\begin{array}{c|cccc}
\ast & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 2 & 2 \\
1 & 1 & 0 & 3 & 2 \\
2 & 2 & 2 & 0 & 0 \\
3 & 3 & 2 & 1 & 0 \\
\end{array}
\quad
\begin{array}{c|cccc}
\cdot & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 \\
2 & 0 & 0 & 2 & 2 \\
3 & 0 & 1 & 2 & 3 \\
\end{array}
\]

Then, by routine calculations, we can see that $(X; \ast, \cdot, 0)$ is a BCI-semigroup.

Example 1.3. Define two binary operations "$\ast$" and "$\cdot$" on a set $X := \{0, a, b, c\}$ as follows:

\[
\begin{array}{c|cccc}
\ast & 0 & a & b & b \\
\hline
0 & 0 & 0 & c & d \\
a & a & 0 & c & d \\
b & b & b & 0 & c \\
c & c & c & b & o \\
\end{array}
\quad
\begin{array}{c|cccc}
\cdot & 0 & a & b & b \\
\hline
0 & 0 & 1 & 0 & 0 \\
a & 0 & 1 & 0 & 0 \\
b & 0 & 1 & b & c \\
c & 0 & 1 & c & b \\
\end{array}
\]

Then it is easy to see that $(X; \ast, \cdot, 0)$ is a BCI-semigroup.

If a BCI-semigroup $X$ contains an element $1_X$ such that $1_X \cdot x = x \cdot 1_X = x$ for all $x \in X$, then $X$ is called a BCI-monoid, and we call $1_X$ the multiplicative identity. If every non-zero element of a BCI-monoid $X$ has a multiplicative inverse, then $X$ is called a BCI-group. In what
follows, for convenience, we shall write the multiplication $x \cdot y$ by $xy$. We give some examples of a $BCI$-semigroup which is a generalization of the ring.

**Example 1.4.** Let $Q$ be the set of all rational numbers. Then $(Q, -, 0)$ is a $BCI$-algebra which is not a $BCK$-algebra, since $0 - x \neq 0$ for any non-zero $x$ in $Q$. It is easily verified that $Q = (Q, -, \cdot, 0, 1)$ is a $BCI$-group, where "·" is the ordinary multiplication on $Q$.

**Proposition 1.5 ([6]).** Let $X$ be a $BCI$-semigroup. Then
(i) $0x = x0 = 0$,
(ii) $x \leq y$ implies that $xz \leq yz$ and $zx \leq zy$, for all $x, y, z \in X$.

**Definition 1.6 ([6]).** A non-empty subset $A$ of a $BCI$-semigroup $X$ is called a left(right) $I$-ideal of $X$ if
(i) $A$ is an ideal of a $BCI$-algebra $X$,
(ii) $x \in X$ and $a \in A$ imply that $xa \in A$ ($ax \in A$). Both left and right $I$-ideal is called two-sided $I$-ideal or simply $I$-ideal.

#### 2. Main Results

In this section, we describe the ideal generated by a non-empty stable set in a $BCI$-group as a simple form, and obtain an equivalent condition of a prime $I$-ideal.

**Theorem 2.1 ([6]).** Let $\{A_i\}$ be a collection of $I$-ideals of the $BCI$-semigroup $X$, where $i$ ranges over some index set. Then $\cap A_i$ is also an $I$-ideal of $X$.

**Definition 2.2.** Let $(X : \ast, \cdot, 0)$ be a $BCI$-semigroup and let $A$ be a subset of $X$. Then the intersection of all $I$-ideals of $X$ containing $A$ is said to be the ideal generated by $A$.

Notice that this definition is well-defined since there is always at least one ideal containing $A$, i.e., $X$ itself. For convenience the ideal generated by $A$ will be denoted by $< A >$. We follow the convention: $\langle \phi \rangle = 0$, and $\langle \{a_1, \ldots, a_n\} \rangle = \langle a_1, \ldots, a_n \rangle$. The elements $a_1, \ldots, a_n$ are said to be the generators of $\langle a_1, \ldots, a_n \rangle$. An ideal $\langle a \rangle$ generated by a single element is called a principal $I$-ideal. A principal $BCI$-semigroup is a $BCI$-semigroup in which every $I$-ideal is principal.
**Definition 2.3.** A non-empty subset $A$ of a semigroup $(X, \cdot)$ is called left (right) stable if for any $x \in X$ and any $a \in A$, $x \cdot a \in A$ ($a \cdot x \in A$). Both left and right stable is two-sided stable or simply stable.

**Example 2.4.** In the Example 1.2, the set $\{0, 1\}$ is stable, while $\{0, 3\}$ is not stable.

**Theorem 2.5.** Let $X$ be a BCI-group and commutative with respect the operation "\(\cdot\)" and $A$ be a non-empty stable subset of $X$. Then

$\langle A \rangle = \{ x \in X \mid \exists a_1, \ldots, a_n \in A \text{ and } \exists r_1, \ldots, r_n \in X - \{0\} \text{ such that } r_n(\cdots(r_2(r_1(x \cdot a_1) \cdot a_2) \cdots) \cdot a_n) = 0 \} \quad (*)$

**Proof.** Denote the right of $(*)$ by $B$. Clearly $0 \in B$. Let $x \cdot y \in B$ and $y \in B$. Then there exist $a_1, \ldots, a_n, b_1, \ldots, b_m \in A$ and $r_1, \ldots, r_n, s_1, \ldots, s_m \in X - \{0\}(n \geq m)$ such that

$r_n(\cdots(r_2(r_1((x \cdot y) \cdot a_1) \cdot a_2) \cdots) \cdot c_n) = 0,$

$s_m(\cdots(s_2(s_1(y \cdot b_1) \cdot b_2) \cdots) \cdot b_m) = 0.$

By the Proposition 1.5-(i), we may assume that $n \geq m$.

So $r_n(\cdots(r_2(r_1(x \cdot a_1) \cdot a_2) \cdots) \cdot a_n) \cdot r_n \cdots r_1 y = 0$, and hence

$r_n(\cdots(r_2(r_1(x \cdot a_1) \cdot a_2) \cdots) \cdot a_n) \leq r_n \cdots r_1 y.$

Leftly "\(\cdot\)"-multiplying both sides of the above inequality by $s_1$, we have

$s_1(r_n(\cdots(r_2(r_1(x \cdot a_1) \cdot a_2) \cdots) \cdot a_n)) \leq s_1 r_n \cdots r_1 y = r_n \cdots r_1 s_1 y.$

Rightly "\(\cdot\)"-multiplying both sides of the above inequality by $s_1 r_n \cdots r_1 b_1$, by Proposition 1.5-(ii), we have

$s_1(r_n(\cdots(r_2(r_1(x \cdot a_1) \cdot a_2) \cdots) \cdot a_n)) \cdot s_1 r_n \cdots r_1 b_1$

\[ \leq r_n \cdots r_1(s_1(y \cdot b_1)) \]

and hence

$s_1(r_n(\cdots(r_2(r_1(x \cdot a_1) \cdot a_2) \cdots) \cdot a_n) \cdot r_n \cdots r_1 b_1) \leq r_n \cdots r_1(s_1(y \cdot b_1)).$
Leftly "\(\cdot\)"-multiplying both sides of the above inequality by \(s_2\), we have

\[
s_2(s_1(r_n(\cdots(r_2(r_1(x*a_1)*a_2)*\cdots)*a_n)*r_n\cdots r_1b_1)) \leq s_2(r_n\cdots r_1(s_1(y*b_1))).
\]

Rightly "\(*\)"-multiplying both sides of the above inequality by \(s_2r_n\cdots r_1b_2\), we have

\[
s_2(s_1(r_n(\cdots(r_2(r_1(x*a_1)*a_2)*\cdots)*a_n)*r_n\cdots r_1b_1) *r_n\cdots r_1b_2) \leq r_n\cdots r_1(s_2(s_1(y*b_1)*b_2)).
\]

Repeating the above argument \(m\)-times we obtain

\[
s_m(\cdots(s_1(r_n(\cdots(r_2(r_1(x*a_1)*a_2)*\cdots)*a_n)*r_n\cdots r_1b_1)*\cdots) *r_n\cdots r_1b_m) \leq r_n\cdots r_1(s_m(\cdots(s_2(s_1(y*b_1)*b_2)*\cdots)*b_m)) = 0.
\]

Consequently,

\[
s_m(\cdots(s_1(r_n(\cdots(r_2(r_1(x*a_1)*a_2)*\cdots) *a_n)*r_n\cdots r_1b_1)*\cdots) *r_n\cdots r_1b_m) = 0.
\]

This implies \(x \in B\).

For any \(k \in X\) and \(x \in B\), there exist \(a_1, \cdots, a_n \in A\) and \(r_1, \cdots, r_n \in X\) such that

\[
r_n(\cdots(r_2(r_1(x*a_1)*a_2)*\cdots) *a_n) = 0.
\]

Since \(A\) is stable, for any \(k \in X\), \(ka_i \in A\) (and \(a_ik \in A\)). So

\[
r_n(\cdots(r_2(r_1(kx*ka_1)*ka_2)*\cdots) *ka_n) = k(r_n(\cdots(r_2(r_1(x*a_1)*a_2)*\cdots) *a_n)) = k \cdot 0 = 0.
\]

Hence \(kx \in B\) (and \(zk \in B\)). Summarizing the above facts \(B\) is an \(I\)-ideal of a \(BCI\)-semigroup \(X\). Obviously, \(A \subseteq B\).
Let $I$ be any $\mathcal{I}$-ideal containing $A$. In order to prove $B \subseteq I$, we assume that $x \in B$. Then there are $a_1, \ldots, a_n \in A$ and $r_1, \ldots, r_n \in X$ such that
\[ r_n(\cdots(r_2(r_1(x*a_1)*a_2)*\cdots)*a_n) = 0. \]
Since $0 \in I$, we have
\[ r_n(\cdots(r_2(r_1(x*a_1)*a_2)*\cdots)*a_n) \in I, \]
so
\[ r_nr_{n-1}(\cdots(r_2(r_1(x*a_1)*a_2)*\cdots)*a_{n-1})*r_na_n \in I. \]
Since $I$ is an $\mathcal{I}$-ideal and $r_na_n \in I$, it follows that
\[ r_nr_{n-1}(\cdots(r_2(r_1(x*a_1)*a_2)*\cdots)*a_{n-1}) \in I. \]
Repeating this argument $n$ times we obtain
\[ r_n \cdots r_1x \in I. \]
Since $X$ is a $BCI$-group, we obtain $x \in I$. Hence $B \subseteq I$ and $B =< A >$, proving the theorem. \hfill \Box

**Definition 2.6.** An $\mathcal{I}$-ideal $P \neq X$ in a $BCI$-semigroup $X$ is said to be prime if it has the following property: If $A$ and $B$ are $\mathcal{I}$-ideals in $X$ such that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$.

**Example 2.7.** In Example 1.2, the set $\{0,1\}$ is prime $\mathcal{I}$-ideal of the $BCI$-semigroup $X$.

**Theorem 2.8.** If $P$ is an $\mathcal{I}$-ideal of a $BCI$-semigroup $X$ such that $P \neq X$ and for all $a, b \in X$
\[ ab \in P \Rightarrow a \in P \quad \text{or} \quad b \in P \quad (**) \]
then $P$ is prime. Conversely, if $(X; *, 0)$ is an associative $BCI$-algebra, $(X, \cdot)$ is a commutative semigroup and the operation "\cdot" is distributive on both side over the operation "\(*\)", then any prime $\mathcal{I}$-ideal $P$ satisfies the condition (**).
Proof. If $A$ and $B$ are $I$-ideals such that $AB \subseteq P$ and $A \not\subseteq P$, then there exists an element $a \in A - P$. Since $ab \in AB \subseteq P$ for any $b \in B$, we have $b \in P$ by applying the condition (**) Hence $B \subseteq P$. This means $P$ is a prime $I$-ideal of $X$.

Conversely, let $P$ be a $I$-ideal of $X$ and $ab \in P$. Then $< ab > \subseteq P$. We claim that $< a > < b > \subseteq < ab >$. Let $x \in < a >$ and $y \in < b >$. Then by Theorem 2.5 there are $r, s \in X - \{0\}$ such that $r(x * a) = 0$ and $s(y * b) = 0$. Hence

$$rs(xy * ab) = rs(xy * ab) * (sb0 * ra0)$$
$$= rs(xy * ab) * (sb * r(x * a) * ra * s(y * b))$$
$$= rs((xy * ab) * rs(b(x * a) * a(y * b)))$$
$$= rs((xy * ab) * ((bx * (ay * ab)) * ba))$$
$$= rs((xy * ab) * ((bx * ay) * (ab * ba)))$$
$$= rs((xy * ab) * (bx * ay))$$
$$= rs(xy * ab) * rs(xb * ay)$$
$$=((rsy * rasb) * rsxb) * rasy$$
$$=((rsy * rasb) * rasy) * rsxb$$
$$=((rsy * rasy) * rasb) * rxsb$$
$$=((rsy * (rasy * rasb)) * rxsb)$$
$$=((x * sy * rx * sb) * (ra * sy * ra * sb))$$
$$= r(x * sb) * ra(sy * sb)$$
$$= (rx * ra) * (sy * sb)$$
$$= 0.$$ 

This means that $xy \in < ab >$. Hence $< a > < b > \subseteq < ab > \subseteq P$. Since $P$ is prime, $< a > \subseteq P$ or $< b > \subseteq P$, whence $a \in P$ or $b \in P$. \[
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Acknowledgements. The authors express their thanks to the referee for his/her valuable suggestions.
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