COMPLEMENTED SUBLATTICES OF $wL_1$

ISOMORPHIC TO CLASSICAL BANACH LATTICES

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Abstract. We investigate complemented Banach subspaces of the Banach envelope of weak$L_1$. In particular, the Banach envelope of weak$L_1$ contains complemented Banach sublattices that are isometrically isomorphic to $l_p, (1 \leq p < \infty)$ or $c_0$. Finally, we also prove that the Banach envelope of weak$L_1$ contains an isomorphic copy of $l^{p, \infty}, (1 < p < \infty)$.

1. Introduction

The space weak$L_1$, as a Lorentz space $L(1, \infty)$, was introduced in analysis because key operators of harmonic analysis do not map $L_1$ into $L_1$. Examples of such operators are the Hardy-Littlewood maximal function and the Hilbert transform. In this viewpoint, it became natural to investigate weak$L_1$, the space of measurable functions $f$ satisfying $\mu(\{x \in \Omega : |f(x)| > y\}) \leq \frac{c}{y}$.

It is known that (except for some trivial measure space), weak$L_1$ is not normable (see [C-S]). The question therefore arise as to whether any nontrivial continuous linear functionals on weak$L_1$ exists. In [C-S], the answer for this question was considered. This implies weak$L_1$ has nontrivial dual space. In [K-P], J. Kupka and T. Peck studied the structure of weak$L_1$. They showed that the space $L_\infty$ is dense in the dual of weak$L_1$ endowed with weak*-topology and showed lattice embeddings of $L_1, l_1[0,1], l_\infty$ and $c_0[0,1]$ into $wL_1$ where $wL_1$ is the Banach envelope of weak$L_1$. Later on, T. Peck and M. Talagrand ([P-T]) proved that every separable order continuous Banach lattice is lattice isometric to

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a sublattice of \( wL_1 \). Finally, H. Lotz and T. Peck ([L-P]) removed the hypothesis of order continuity in the separable case.

As a Lorentz space, we will study the space \( L(\cdot, \infty) \) which is called \( \textit{weak}L_1 \) denoted \( wL_1 \):

\[
(1.1) \quad wL_1 = \{ f \in L_0 : \mu(\{ x \in \Omega : |f(x)| > y \}) < \frac{c}{y} \},
\]

where \( c > 0 \) is independent of \( y > 0 \). As we mentioned in the above, \( wL_1 \) is not normable, but we can find nontrivial linear functionals on \( wL_1 \). This was first observed by M. Cwikel and Y. Sagher in [C-S].

For \( 0 < p < \infty \), the space \( \textit{weak}L_p \) taken over the measure space \( (\Omega, \Sigma, \mu) \) consists of all (equivalence classes of) measurable functions \( f \) for which the quasinorm

\[
(1.2) \quad q_p(f) = \sup_{a > 0} a \mu(\{ x \in \Omega : |f(x)| > a \})^\frac{1}{p}.
\]

Define \( q \) to be the Minkowski functional of the convex hull of the unit ball \( \{ f \in wL_1 : q_1(f) \leq 1 \} \) of \( wL_1 \) where \( q_1(f) = \sup_{a > 0} a \mu(\{ x \in \Omega : |f(x)| > a \}) \). The fact that \( q \) is a seminorm on \( wL_1 \) is most readily seen from the alternate formulation

\[
q(f) = \inf_{f = f_1 + \cdots + f_n} \sum_{i=1}^{n} q_1(f_i),
\]

where the infimum is taken over all finite decompositions \( f = f_1 + f_2 + \cdots + f_n \) of \( f \) in \( wL_1 \). This gives the Banach envelope seminorm on \( wL_1 \). In [C-F1], if \( \mu \) is nonatomic, then we can get an equivalent integral-like seminorm

\[
(1.3) \quad \| f \|_{wL_1} = \lim_{n \to \infty} \sup_{\frac{1}{p} \geq n} \frac{1}{\log^{\frac{1}{p}} \frac{q}{p}} \int_{\{ p \leq |f| \leq q \}} |f| d\mu.
\]

Later on, in [C-F2] actually the Banach envelope seminorm on \( wL_1 \) was calculated to be exactly as above. Note that the seminorm on \( wL_1 \) defined in (1.3) is a lattice seminorm. This is not quite obvious, but using
integration by parts one can readily show that the seminorm \( \| \cdot \|_{wL_i} \) is exactly same as (see [K-P]),

\[
(1.4) \quad \lim_{n \to -\infty} \sup_{\xi \geq n} \frac{1}{\log^q \xi} \int_p^q \mu(\{x \in \mu : |f(x)| > t\}) dt.
\]

Even though \( wL_1 \) is complete with respect to the quasinorm \( q_1 \), it is not complete with respect to the seminorm \( \| \cdot \|_{wL_i} \). This is due to M. Cwikel and C. Fefferman ([C-F1] and [K-P]). Let \( \mathcal{N} = \{ f \in wL_1 : \|f\|_{wL_i} = 0 \} \). Then we obtain the quotient space \( wL_1/\mathcal{N} \). We define \( wL_1 \) as the normed envelope (and its completion as the Banach envelope) of \( wL_1 \).

To study this subject, we need some basic facts about the dual of \( wL_1 \). We would like to convert the nonlinear limit superior expression (1.4) for \( \| \cdot \|_{wL_i} \) into a linear limit expression by directing the numbers \( I^b_a(f) = \frac{1}{\log \frac{b}{a}} \int_{\{a \leq |f| \leq b\}} |f| d\mu \) in some fashion. For this, we introduce an ultrafilter \( \mathcal{U} \) so that the limit of the \( I^b_a \) along \( \mathcal{U} \) determines a canonical integral-like linear functional \( I_\mathcal{U} \in wL_1^* \). We now begin with the discussion of \( \mathcal{U} \). For \( n = 1, 2, 3, \ldots \), let

\[
(1.5) \quad F_n = \{(a, b) : 1 \leq a < b, \frac{b}{a} \geq n\}.
\]

and then define \( \mathcal{F} = \{F_n : n \geq 1\} \). Treating \( \mathcal{F} \) as a filter of subsets of the set \( S = [1, \infty) \times [1, \infty) \), we obtain from Zorn's lemma an ultrafilter \( \mathcal{U} \) of subsets of \( S \) such that \( \mathcal{F} \subset \mathcal{U} \).

From now on, we will fix the ultrafilter \( \mathcal{F} \subset \mathcal{U} \). Define the "ersatz integral" \( I_\mathcal{U} \) for every nonnegative function \( f \in wL_1 \) by

\[
(1.6) \quad I_\mathcal{U}(f) = \lim_{\mathcal{U}} I^b_a(f) = \lim_{\mathcal{U}} \frac{1}{\log \frac{b}{a}} \int_{\{a \leq f \leq b\}} f d\mu.
\]

**Theorem 1.1 (J. Kupka and T. Peck).** Let \( f, g \in wL_1 \) be nonnegative and let \( r > 0 \). Then we have

i) \( I_\mathcal{U}(rf) = rI_\mathcal{U}(f) \).

ii) \( I_\mathcal{U}(f + g) = I_\mathcal{U}(f) + I_\mathcal{U}(g) \).

iii) If \( f \leq g \), then \( I_\mathcal{U}(f) \leq I_\mathcal{U}(g) \).
iv) \( I_u(f) \leq \|f\|_{wL_1} \).

From these properties, we define \( I_u(f) \) for an arbitrary function \( f \in wL_1 \) by
\[
I_u(f) = \lim_{U} \frac{1}{\log x} \int_{\{a \leq \vert f \vert \leq b\}} (f^+ - f^-) d\mu;
\]
i) \( I_u \) is linear.

ii) \( |I_u(f)| \leq \|f\|_{wL_1} \) for all \( f \in wL_1 \).

iii) \( I_u \) vanishes on \( N = \{ f \in wL_1 : \|f\|_{wL_1} = 0 \} \) and hence determines a well defined, bounded linear functional on \( wL_1 \).

Similarly, N.J. Kalton in [KAL] gave a linear functional on \( wL_1 \) in the following way; for \( f \in L_0 \) and \( x \geq 0 \), we define the truncation \( \tau_x f \) by
\[
\tau_x f(\omega) = f(\omega) \quad \text{if} \quad |f(\omega)| \leq x
\]
\[
= x \quad \text{if} \quad f(\omega) > x
\]
\[
= -x \quad \text{if} \quad f(\omega) < -x.
\]

Then a linear functional on \( wL_1 \) is defined by
\[
\phi(f) = \lim_{U} \frac{1}{\log x} \mathcal{E}(\tau_x f),
\]
where \( U \) is any ultrafilter on \((2, \infty)\) which includes each of the sets \((x, \infty)\) for \( x > 2 \) and \( \mathcal{E} \) is expectation.

We now have information on the dual of \( wL_1 \):

**Theorem 1.2 (J. Kupka and T. Peck).** Define a linear operator \( T_u : L_\infty \rightarrow wL_1^* \) by \( T_u(m) : f \mapsto I_u-mf \) for all \( m \in L_\infty(\mu) \), and for all \( f \in wL_1 \). Then \( T_u \) constitutes an isometric, order isomorphism of \( L_\infty(\mu) \) into \( wL_1^* \).
Moreover, the linear span of the subspace \( T_u(L_\infty(\mu)) \), as \( U \) ranges over the collection of ultrafilters which contains \( \mathcal{F} \), constitutes a norming and hence a weak* dense subspace of \( wL_1^* \).

We now are in a position to prove the results. For the proof of those, we need several lemmas. Note that if \( f, g \in uL_1 \), \( |f| \wedge |g| = 0 \) with \( \|f\|_{wL_1} = \|g\|_{wL_1} = 1 \), then by the Hahn-Banach theorem, we
can find $\phi, \psi \in wL_1^*$ with $\|\phi\| = \|\psi\| = 1$, $\phi(f) = \psi(g) = 1$ and $\phi(g) = \psi(f) = 0$. Then for arbitrary $h \in wL_1$, we have

$$|\phi(h) + \psi(h)| \leq |\phi(h)| + |\psi(h)|$$

$$\leq 2\|h\|_{wL_1}.$$ 

This estimate is not good enough for our purpose.

This theorem gives some favorable information for $wL_1(U)^*$ and the very last part of theorem says

$$T_U(B_{L_\infty})^{wL_1(U)^*} = B_{wL_1(U)^*}.$$

From this theorem, for any $m \in L_\infty(\mu)$, we have

$$\widehat{m} = T_U(m) \in wL_1(U)^*.$$  

Clearly, every linear functional $\varphi \in wL_1(U)^*$ is a linear functional on $wL_1$ (see more detail in [K-P, 2.20]).

We now give a lemma about linear functionals on $wL_1$ which is actually due to J. Kupka and T. Peck (see [K-P.2.20]).

**Lemma 1.3.** For a fixed ultrafilter $U$ in (1.5), let $f \in wL_1$ be a nonnegative function with $\|f\|_U = 1$. Then for any $g \in wL_1$, disjointly supported from $f$, there exists a $\phi \in wL_1^*$ such that $\|\phi\| = 1$, $\phi(f) = 1$, and $\phi(g) = 0$.

We can now generalize this lemma for arbitrary pairwise disjointly supported elements in $wL_1$.

**Corollary 1.4.** Let $(f_n)_{n=1}^\infty$ be a sequence of nonnegative elements in $wL_1$ with $\|f_n\|_{wL_1} = 1$, for all $n = 1, 2, 3, \ldots$, and such that the $f_n$ have pairwise disjoint supports. Then for each $n$, there exists a linear functional $\phi_n$ on $wL_1$ such that $\phi_n(f) = 1$, $\|\phi_n\| = 1$ and $\phi_n(f_m) = 0$ if $n \neq m$.

**Proof.** We can show this by an induction with Lemma 1.3 for each $f_n$. For given $f_1 \in wL_1$, by Lemma 1.3, we can choose $\phi_1$ with $\phi_1(f_1) = \|\phi_1\| = 1$ and $\phi_1(f_j) = 0$, for all $j = 2, 3, \ldots$. If we selected $\phi_1, \phi_2, \ldots, \phi_n$ satisfying all the conclusions of corollary, then $\phi_{n+1}$ can be selected by applying Lemma 1.3 again. This proves the corollary.
Remark 1.5. Define \( wL_1(\mathcal{U}) = \{ f \in wL_1 : \| f \|_u < \infty \} \) where \( \mathcal{U} \) is the ultrafilter defined in (1.1). Then we have \( \| f \|_u \leq \| f \|_{wL_1} \) where \( \| f \|_u = I_\mathcal{U}(|f|) = \lim u \frac{1}{\log \frac{1}{2}} \int_{\{a \leq |f| \leq b\}} |f| d\mu \). Hence we have \( wL_1 \subseteq wL_1(\mathcal{U}) \). Moreover \( \| f \|_u = I_\mathcal{U}(f) \) has the following properties:

i) \( \| \cdot \|_u \) is a lattice seminorm on \( wL_1 \),

ii) \( \| f + g \|_u = \| f \|_u + \| g \|_u \) whenever \( f \) and \( g \) are nonnegative,

iii) \( \| f \|_{wL_1} = \sup \{ \| f \|_u : \mathcal{U} \text{ is an ultrafilter, } \mathcal{F} \subseteq \mathcal{U} \} \) for all \( f \in wL_1 \).

Again, we convert \( \| \cdot \|_u \) into a norm by forming the ideal

(1.10) \[ \mathcal{N}_u = \{ f \in wL_1 : \| f \|_u = 0 \}. \]

and then the quotient vector lattice \( wL_1(\mathcal{U}) = wL_1/\mathcal{N}(\mathcal{U}) \) on which \( \| \cdot \|_u \) acts as a lattice norm. We need one technical lemma, namely;

Lemma 1.6. Let \( (f_n)_{n=1}^\infty \) be a sequence of nonnegative elements in \( wL_1 \) such that the \( f_n \) have pairwise disjoint supports with \( \| f_n \|_{wL_1} = 1 \) for all \( n = 1, 2, 3, \ldots \) and let \( (\phi_n)_{n=1}^\infty \) be a sequence of linear functionals on \( wL_1 \) selected as in Corollary 1.4. Then for any \( f \in wL_1 \), we have

\[ \sum_{n=1}^\infty |\phi_n(f)| \leq \| f \|_{wL_1}. \]

Proof. For an arbitrary function \( f \in wL_1 \), the number \( \phi_n(f) \) is the limit of a subnet of the sequence \( \{ I_\mathcal{U}(\chi_{E_n,k} \cdot f) \} \) where \( (E_{n,k})_{k=1}^\infty \) is a decreasing sequence of subsets of \( E_n = \text{supp}(f_n) \), and \( f_n \) is bounded on \( E_{n,k} \) for all \( k \) (see Corollary 1.4). Fix \( n \neq m \), let \( (E_{n,k})_{k=1}^\infty \) be the decreasing sequence of measurable sets for \( f_n \) and \( (E_{m,k})_{k=1}^\infty \) the corresponding sequence for \( f_m \).

Let \( r = \text{sgn} I_\mathcal{U}(\chi_{E_{n,k}} \cdot f), \quad s = \text{sgn} I_\mathcal{U}(\chi_{E_{m,k}} \cdot f) \). Put \( m = r \chi_{E_{n,k}} + s \chi_{E_{m,k}} \) so that \( \| m \|_\infty = 1 \). By (1.10) we have that for \( m \in L_\infty, \tilde{m} \in wL_1 \). Then we have

(1.11) \[ \tilde{m}(f) = |I_\mathcal{U}(\chi_{E_{n,k}} \cdot f)| + |I_\mathcal{U}(\chi_{E_{m,k}} \cdot f)| \]

\[ = I_\mathcal{U}(m \cdot f) \]

\[ \leq \| m \|_\infty \| f \|_u \quad \text{since} \quad \| m \|_\infty = 1 \]

\[ = \| f \|_u \]

\[ \leq \| f \|_{wL_1}. \]
By the additive rule for nets, we have that in the limit

\[(1.12) \quad |\phi_n(f)| + |\phi_m(f)| \leq \|f\|_u \]
\[\leq \|f\|_{wL_1} .\]

To show \(\sum_{n=1}^{\infty} |\phi_n(f)| \leq \|f\|_{wL_1}\), it suffices to show that for any \(N \in \mathbb{N}\), \(\sum_{n=1}^{N} |\phi_n(f)| \leq \|f\|_{wL_1}\). For \(n = 1, 2, 3, \ldots\), let \((E_{n,k})_{k=1}^{\infty}\) be the decreasing sequence of measurable sets for \(f_n\) and \(E_n = \text{supp}(f_n)\). Let \(r_n = \text{sgn}I_U(\chi_{E_{n,k}} \cdot f)\). Put \(m = \sum_{n=1}^{N} r_n \chi_{E_{n,k}}\). Then we have \(\|m\|_\infty = 1\). By the same argument as the above, we have

\[(1.13) \quad \hat{m}(f) = \sum_{m=1}^{N} |I_U(\chi_{m,k} \cdot f)| \]
\[= I_U(mf) \]
\[\leq \|m\|_\infty \|f\|_u \]
\[\leq \|f\|_{wL_1} .\]

By the additive rule for nets, we have

\[(1.14) \quad \sum_{n=1}^{N} |\phi_n(f)| \leq \|f\|_u \]
\[\leq \|f\|_{wL_1} .\]

We can therefore have that \(\sum_{n=1}^{\infty} |\phi_n(f)| \leq \|f\|_{wL_1}\). This proves the lemma.

2. **Isomorphic copies of \(l_p, (1 \leq p < \infty)\) in \(wL_1\).**

We will investigate a complemented sublattice of \(wL_1\) that is isometrically isomorphic to \(l_p\) for \((1 \leq p < \infty)\). The space \(l_p\) is a classical Banach lattice which is separable and order continuous. Also the usual basis \((e_i)_{i=1}^{\infty}\) is an unconditional basis for \(l_p\). To prove this, we need a theorem which was done by H. Lotz and T. Peck.
THEOREM 2.1 (H. Lotz and T. Peck). Let \( E \) be a separable Banach lattice. Then \( E \) is lattice isometric to a closed sublattice of the Banach envelope of weak\( L_1 \).

PROPOSITION 2.2. For \( 1 \leq p < \infty \), there is a lattice isometry \( T \) from \( l_p \) to \( wL_1 \), that is, \( l_p \) can be embedded isometrically into \( wL_1 \).

PROOF. For \( 1 \leq p < \infty \), \( l_p \) is a separable Banach lattice with order continuous norm. Hence by Theorem 2.1, there exists a lattice isometry \( T \) from \( l_p \) into \( wL_1 \). Then \( T \) is the desired lattice isometric embedding linear map. This proves the proposition.

REMARK 2.3. In [K-P], J. Kupka and T. Peck proved that there exists an isometric, order isomorphism \( T \) from \( l_1[0,1] \) into \( wL_1 \). Moreover, the range of \( T \) is a complemented subspace of \( wL_1 \).

Thank to Proposition 2.2, we can embed \( l_p \) into \( wL_1 \) under a lattice isometry. So one can ask the natural question: What can we say about the range of \( T \) for \( l_p, \ 1 \leq p < \infty \) in Proposition 2.2? We can now answer this as follows:

THEOREM 2.4. For \( 1 \leq p < \infty \), let \( T : l_p \to wL_1 \) be a lattice isometry given by \( T(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i=1}^{\infty} a_i f_i \) where \( (e_i)_{i=1}^{\infty} \) is the usual basis of \( l_p \). Then the range of \( T \) is a complemented sublattice of \( wL_1 \).

PROOF. Let \( T : l_p \to wL_1 \) be a lattice isometry, and let \( (e_i)_{i=1}^{\infty} \) be the usual basis for \( l_p \). Define \( Te_i = f_i, \ i = 1, 2, 3, \ldots \). Then since \( T \) preserves lattice structure, \( f_i \) are nonnegative pairwise disjoint and \( \|f_i\|_{wL_1} = 1 \), for all \( i = 1, 2, 3, \ldots \). Now, by Corollary 1.4, we can find linear functionals \( \phi_n \) on \( wL_1 \) such that \( \phi_n(f_n) = 1, \ \phi_n(f_m) = 0 \) if \( n \neq m \) and \( \|\phi_n\| = 1 \), for all \( n = 1, 2, 3, \ldots \). For an arbitrary function \( f \in wL_1 \), the number \( \phi_n(f) \) is the limit of a subnet of the sequence \( \{I_{U}(\chi_{E_{n,k} \cap f})\} \) where \( (E_{n,k})_{k=1}^{\infty} \) is a decreasing sequence of subsets of \( E_n = \text{supp}(f_n) \) and \( f_n \) is bounded on \( E_{n,k}^c \) for all \( k \) and for all \( n = 1, 2, 3, \ldots \).

Now, we can define a contractive projective \( P \) from \( wL_1 \) onto \( Tl_p \),

\[
P(f) = \sum_{n=1}^{\infty} \phi_n(f) f_n.
\]

(2.1)
First, we need to show that $P$ is well defined. Since $(f_n)_{n=1}^\infty$ is a copy of the usual $l_p$ basis in $wL_1$, we have for $1 \leq p < \infty$,

\[(2.2) \quad \| \sum_{n=1}^\infty \phi_n(f) f_n \|_{wL_1} = (\sum_{n=1}^\infty |\phi_n(f)|^p)^{\frac{1}{p}} \leq \sum_{n=1}^\infty |\phi_n(f)| \leq \|f\|_U \quad \text{(by Lemma 1.6)} \leq \|f\|_{wL_1}.
\]

Hence, $P(f) = \sum_{n=1}^\infty \phi_n(f) f_n \in Tl_p$, for all $f \in wL_1$.

Next we need to show that $\|P\| = 1$. By (2.2), we have $\|P(f)\|_{wL_1} \leq \|f\|_{wL_1}$. Hence $\|P\| \leq 1$. On the other hand, since $f_m \in Tl_p \subset wL_1$, we have

\[(2.3) \quad P(f_m) = \sum_{n=1}^\infty \phi_n(f_m) f_n \]

\[= \phi_m(f_m) f_m \]

\[= f_m.
\]

Therefore we have $\|P(f_m)\|_{wL_1} = \|f_m\|_{wL_1} = 1$. This shows $\|P\| = 1$.

Finally, we need to show that $P^2 = P$. For $f \in wL_1$,

\[(2.4) \quad P^2(f) = P\left( \sum_{n=1}^\infty \phi_n(f) f_n \right) \]

\[= \sum_{j=1}^\infty \phi_j \left( \sum_{n=1}^\infty \phi_n(f) f_n \right) f_j \quad \text{by} \quad \delta_n(f_j) = \delta_{n,j} \]

\[= \sum_{j=1}^\infty \phi_j(f) f_j \]

\[= P(f).
\]

Therefore $P$ is a norm one projection from $wL_1$ onto $Tl_p$. This proves the theorem.

Now, we can have the main result.
COROLLARY 2.5. For $1 \leq p < \infty$, the Banach envelope of $wL_1$ contains a complemented subspace that is isometrically isomorphic to $l_p$.

PROOF. Immediate from Theorem 2.4.

3. An isomorphic copy of $c_0$ in $wL_1$.

The Banach space $c_0 = \{(a_n)_{n=1}^{\infty} : \lim_{n \to \infty} a_n = 0\}$ is a separable space with basis $(e_n)_{n=1}^{\infty}$ where $e_i = (0, 0, 0, \ldots, 1_i, 0, \ldots)$. The space $c_0$ is not reflexive, but is $\sigma$-complete, and a $\sigma$-order continuous Banach lattice, which means $c_0$ is order continuous. As a universal Banach space for separable Banach spaces $C([0, 1])$ contains a complemented subspace that is isomorphic to $c_0$. The goal of this section is to prove the Banach envelope $wL_1$ contains a complemented subspace that is isomorphic to $c_0$. In [K-P], J. Kupka and T. Peck proved that $c_0[0, 1]$ can be embedded lattice isometrically into $wL_1$. Since we need only a countable index set, we give a modification of [K-P, Theorem 4.4].

PROPOSITION 3.1. There is an isometric, order isomorphic linear embedding of the space $c_0$ into $wL_1$.

PROOF. Let $N = \bigcup_{i=1}^{\infty} S_i$ where the $S_i$ are infinite and pairwise disjoint subsets of $N$. Let $h \in wL_1$, $h \geq 0$, and $\|h\|_{wL_1} = 1$. Then by definition of $\| \cdot \|_{wL_1}$, there exist pairwise disjoint closed intervals $\{[a_n, b_n]\}_{n=1}^{\infty}$ such that $\frac{b_n}{a_n} \to \infty$, $a_n \to \infty$ and such that

$$I_{a_n}^{b_n}(h) = \frac{1}{\log \frac{b_n}{a_n}} \int_{\{a_n \leq h \leq b_n\}} h d\mu$$

$$\geq \|h\|_{wL_1} - \frac{1}{n}$$

for all $n$.

Define for $i \in N$, $B_i = \bigcup_{n \in S_i} [a_n, b_n]$. Now, define $T : c_0 \to wL_1$ by

$$T((a_n)) = \text{pointwise - } \sum_{n=1}^{\infty} a_n h \chi_{B_n}(h), \text{ for } (a_n) \in c_0.$$
where \( \chi_{B_n} (h) \) is the composition of \( \chi_{B_n} \) and \( h \). Then we need to check \( T \) is well defined. Since \( a_n \to 0 \) as \( n \to \infty \), it follows that given \( \epsilon > 0 \), there exists \( M > 0 \) such that if \( n \geq M \) then \( |a_n| < \epsilon \). Then we have

\[
(3.3) \quad \left\| \sum_{M+1}^\infty a_n h \chi_{B_n} (h) \right\|_{wL_1} \leq \max_{n \geq M+1} |a_n| < \epsilon.
\]

Hence \( T \) is well defined.

Clearly, \( T \) is linear and preserves the lattice operations. Finally, we need to check that it is an isometry:

\[
(3.4) \quad \|T(a_n)\|_{wL_1} = \left\| \sum_{n=1}^\infty a_n h \chi_{B_n} (h) \right\|_{wL_1} = \max_n |a_n| \quad \text{(by (3.1))}
\]

\[
= \| (a_n) \|.
\]

Therefore \( T \) is a linear and lattice isometry. This proves the proposition.

For the proof of main result, we need one technical lemma.

**Lemma 3.2.** In the proof of Proposition 3.1, there is a \( h \in wL_1 \) with \( \|h\|_{wL_1} = 1 \) as a strictly increasing nonnegative function with no atoms.

**Proof.** Let \( h \in wL_1, \|h\|_{wL_1} = 1 \) and \( h \geq 0 \). If the function \( h \) satisfies \( \mu(h = r) > 0 \) for some \( r \), then we embed the measure algebra of Lebesgue measure \( \lambda \) on \([0,1]\) into the measure algebra of normalized \( \mu \)-measure on the measurable set \( \{h = r\} \). We replace \( h \) on this set by the image of the function \( \psi(t) = t + r \). Since there are at most countably many points \( r \geq 0 \) for which \( \mu\{h = r\} > 0 \), the performance of such replacement for each of these points will change \( h \) by at most a bounded measurable function. This means that \( h \) can be everywhere strictly positive. This proves the lemma.
Theorem 3.3. Let \( T : c_0 \to wL_1 \) be a lattice isometry given by

\[
T(a_n) = \text{pointwise} - \sum_{n=1}^{\infty} a_n h \chi_{B_n}(h), \quad \text{as in Proposition 3.1.}
\]

Then the range of \( T \) is a complemented sublattice of \( wL_1 \).

Proof. By Lemma 3.2, we can assume that \( h \) is everywhere strictly positive with \( \|h\|_{wL_1} = 1 \). Considering the proof of Proposition 3.1, we defined \( T : c_0 \to wL_1 \) by \( T((a_n)) = \sum_{n=1}^{\infty} a_n h \chi_{B_n} \). Now let \( h_n = h \chi_{B_n}(h) \), where \( B_n = \bigcup_{i \in S_n} [a_i, b_i] \). Then we have \( (h_n)_{n=1}^{\infty} \) pairwise disjointly supported nonnegative functions in \( wL_1 \) with \( \|h_n\|_{wL_1} = 1 \), for all \( n = 1, 2, 3, \ldots \). By the Corollary 1.4 we can find linear functionals \( (\phi_n) \) on \( wL_1 \) such that \( \phi_n(h_m) = \delta_{n,m} \) and \( \|\phi_n\| = 1 \). Now for arbitrary \( f \in wL_1 \), the number \( \phi(f) \) is the limit of a subnet of the sequence \( \{I_U(\chi_{D_{n,k}} \cdot f)\} \), where \( (D_{n,k})_{k=1}^{\infty} \) is a decreasing sequence of subsets of \( D_n = \text{supp}(h_n) \) and \( h_n \) is bounded on \( D_{n,k} \), for all \( k \). For fixed \( n \), let \( r = \text{sgn} I_U(\chi_{D_{n,k}} \cdot f) \). Put \( m = r \chi_{D_{n,k}} \). Then we have \( \|m\|_{\infty} = 1 \). Moreover,

\[
|I_U(\chi_{D_{n,k}} \cdot f)| = \hat{m}(f) = I_U(mf) \\
\leq \|m\|_{\infty} \|f\|_U \\
= \|f\|_U \\
\leq \|f\|_{wL_1}.
\]

Now define a projection \( P : wL_1 \to Tc_0 \) by

\[
P(f) = \sum_{n=1}^{\infty} \phi_n(f) h_n.
\]

First, we need to show that this is well defined. It suffices to show that for all \( f \in wL_1 \), the sequence \( (|\phi_n(f)|)_{n=1}^{\infty} \) converges to zero (so that \( (|\phi_n(f)|) \in c_0 \)). Now
\[ \| \sum_{n=1}^{\infty} \phi_n(f) h_n \|_{wL_1} \leq \sum_{n=1}^{\infty} |\phi_n(f)| \| h_n \|_{wL_1} \]
\[ = \sum_{n=1}^{\infty} |\phi_n(f)| \quad (\text{since } \| h_n \|_{wL_1} = 1, \ n = 1, 2, \ldots) \]
\[ \leq \| f \|_{U} \quad (\text{by Lemma 1.6}) \]
\[ \leq \| f \|_{wL_1}, \]

hence we have \( \phi_n(f) \to 0 \) as \( n \to \infty \), for all \( f \in wL_1 \). Therefore \( P \) is well defined. Moreover we have \( \| P(f) \|_{wL_1} \leq \| f \|_{wL_1} \). This implies \( \| P \| \leq 1 \).

On the other hand, if we take \( h_m \in T(c_0) \subset wL_1 \), then \( P(h_m) = \sum_{n=1}^{\infty} \phi_n(h_m)h_n = \phi_m(h_m)h_m = h_m \). Then we have \( \| P(h_m) \|_{wL_1} = 1 \).

Hence \( P \) is a norm one onto map. The proof of \( P^2 = P \) is just routine. Therefore \( P \) is the desired norm one projection from \( wL_1 \) onto \( T(c_0) \).

This proves the theorem.

Now we can have the main result.

**Corollary 3.4.** The Banach envelope of \( wL_1 \) contains a complemented subspace that is isometrically isomorphic to \( c_0 \).

**Proof.** Immediate from Theorem 3.3.

4. A lattice isomorphic copy of \( l^{p,\infty} \) in \( wL_1 \)

For \( 1 < p < \infty \), the weak \( L_p \) space is the space of all \( \Sigma \)-measurable functions \( f \) such that \( \{ \omega \in \Omega : |f(\omega)| > 0 \} \) is \( \sigma \)-finite and

\[ \| f \| = \sup_{B} \frac{1}{\mu(B)^{\frac{1}{p}}} \int_{B} |f| d\mu < \infty. \]

where the supremum is taken over all measurable sets \( B \) with \( 0 < \mu(B) < \infty \). It is well known that the expression

\[ \| |f| \| = \sup_{t>0} \mu(\{ |f| > t \})^{\frac{1}{p}} \]

is equivalent to the norm \( \| \cdot \| \) defined in (4.1).

In this section, we want to show the existence of a copy of \( l^{p,\infty} \) in \( wL_1 \).
Theorem 4.1 (H.P. Lotz and T. Peck). Let $(\Omega, \Sigma, \mu)$ be a separable measure space and let $1 < p < \infty$. Then $\text{weak}L_p$ is lattice isometric to a closed sublattice of $wL_1$.

First, we want to embed $l^{p,\infty}$ into $wL_1$. If we use Theorem 4.1, it is an easy task because the $l^{p,\infty}$ space is a separable Banach space. Now let $T$ be a lattice isometry given by Theorem 4.1. Does the range of $T$ a complemented sublattice in $wL_1$? We do not know whether the range of $T$ is complemented. But if we use another lattice isomorphism, we can have that the range of the lattice embedding map is a complement sublattice of $wL_1$. Hence we will use another embedding of $l^{p,\infty}$ into $wL_1$.

Now, referring to [LEU], fix $p \ (1 < p < \infty)$ and let $q = \frac{p}{p-1}$. For $n \geq 3$, let

\[ g_n = \sup_{j \geq n} (j!)^\frac{1}{p} \lambda A_j^n, \]

where $A_j^n = [2^{1-n} + \sum_{m=j+1}^{\infty} \frac{1}{m!}, 2^{1-n} + \sum_{m=j}^{\infty} \frac{1}{m!})$.

Then $(g_n)$ is a pairwise disjoint sequence of measurable functions on $[0,1]$.

Theorem 4.2 (Denny H. Leung). The map $T : l^{p,\infty} \rightarrow \text{weak}L_p[0,1]$ defined by $T((a_n)_{n=3}^{\infty}) = \text{pointwise} - \sum_{n=3}^{\infty} a_n g_n$ is a lattice isomorphism.

It is not hard to see that $T$ is not a lattice isometry (see [LEU]).

Proposition 4.3. For $1 < p < \infty$, the $l^{p,\infty}$ space is lattice isomorphic to a closed sublattice of $wL_1$.

Proof. Immediate from Theorem 4.1 and Theorem 4.2.

Let $U : l^{p,\infty} \rightarrow wL_1$ be a lattice isomorphism with $\|U\| = C < \infty$. Then $U$ preserves the lattice structure of $l^{p,\infty}$ in $wL_1$. Let $S : \text{weak}L_p[0,1] \rightarrow wL_1$ be a lattice isometry given by Theorem 4.1. Then we can define $S(g_n) = f_n$ for each $n \geq 3$. Since $(g_n)$ are pairwise disjoint, $(f_n)$ are also pairwise disjoint. Now we can have the main result.
\textbf{Theorem 4.4.} For $1 < p < \infty$, the Banach envelope of $wL_1$ contains a complemented sublattice which is isomorphic to $l^p, \infty$.

\textbf{Proof.} Define $U : [l^p, \infty] \rightarrow wL_1$ by

$$U((a_n)_{n=3}^\infty) = \sum_{n=3}^\infty a_n f_n,$$

for each $((a_n)_{n=3}^\infty) \in [l^p, \infty]$.

where $f_n = Sg_n$ for each $n \geq 3$ and $g_n$ is defined in Theorem 4.2. Without loss of generality, we can assume $\|f_n\|_{wL_1} = 1$, by normalizing. It suffices to show that the range of $U$ is complemented in $wL_1$. Since $(f_n)$ are pairwise disjoint and $\|f_n\|_{wL_1} = 1$ for each $n \geq 3$, we can find linear functionals $\phi_n \in wL_1^*$ with $\phi_n(f_m) = \delta_{n,m}$, and $\|\phi_n\| = 1$, for each $n = 3, 4, \ldots$ by Corollary 1.4.

Define $P : wL_1 \rightarrow U[l^p, \infty]$ by

$$P(f) = \sum_{n=3}^\infty \phi_n(f) f_n.$$

We need to show that $P$ is well defined.

$$\|P(f)\|_{wL_1} = \|\sum_{n=3}^\infty \phi_n(f) f_n\|_{wL_1}$$

$$= \|\sum_{n=3}^\infty \phi_n(f) g_n\|,$$

where $Sg_n = f_n$.

$$\leq C \|\|(\phi_n(f))\||\|,$$

where $C$ is an isomorphism constant.

$$\leq D \sup_B \frac{1}{\mu(B)^{\frac{1}{p}} - \frac{1}{p}} \sum_{n \in B} |\phi_n(f)|$$

$$\leq D \sum_{n=3}^\infty |\phi_n(f)|$$

$$\leq D \|f\|_{wL_1} \quad \text{(by Lemma 1.6)},$$

where the equality follows from the fact that $(a_n)_{n=3}^\infty \mapsto \sum_{n=3}^\infty a_n g_n$ is an isomorphic lattice embedding, the first inequality comes from the equivalence
of the quasinorm $|| \cdot ||$ defined in (4.3) and the norm $|| \cdot ||$ defined in (4.2), and the second inequality follows from the fact that $\mu(B) \geq 1$ and $\mu$ in counting measure. Therefore $||P|| \leq D$ and $P$ is well defined.

The proofs of linearity and $P^2 = P$ are just routine. Therefore $U_{L^p, \infty}$ is a complemented sublattice of $wL_1$. This proves the theorem.

References


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