THE N-TH PRETOPOLOGICAL MODIFICATION OF CONVERGENCE SPACES

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Abstract. In this paper, we introduce the notion of the n-th pretopological modification. Also, we find some properties which hold between convergence quotient maps and n-th pretopological modifications.

1. Introduction

A convergence structure defined by Kent [4] is a correspondence between the filters on a given set \( X \) and the subsets of \( X \) which specifies which filters converge to points of \( X \). This concept is defined to include types of convergence which are more general than that defined by specifying a topology on \( X \). Thus, a convergence structure may be regarded as a generalization of a topology.

With a given convergence structure \( q \) on a set \( X \), Kent [4] introduced an associated convergence structure which is called a pretopological modification.

Also, Kent [6] introduced a convergence quotient map, which is a quotient map for a convergence space.

In this paper, with a convergence structure \( q \), we introduce notions of the filter \( V_q^n(x) \) and the n-th pretopological modification of \( q \) which is denoted by \( \pi_n(q) \), where \( n \in \mathbb{N} \cup \{\infty\} \).

In Theorem 7, we show that for a map \( f: (X, q) \to (Y, p) \), \( V_p(f(x)) = f(V_q(x)) \) iff \( V_p^n(f(x)) = f(V_q^n(x)) \) for each \( n \in \mathbb{N} \cup \{\infty\} \).

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In Theorem 10, we show that if $p$ is pretopological and $f: (X, q) \to (Y, p)$ is a convergence quotient map, then $f: (X, \pi_n(q)) \to (Y, \pi_n(p))$ is also a convergence quotient map for each $n \in N \cup \{\infty\}$.

2. Preliminaries

A convergence structure $q$ on a set $X$ is defined to be a function from the set $F(X)$ of all filters on $X$ into the set $P(X)$ of all subsets of $X$, satisfying the following conditions:

1. $x \in q(\hat{x})$ for all $x \in X$;
2. $\Phi \subseteq \Psi$ implies $q(\Phi) \subseteq q(\Psi)$;
3. $x \in q(\Phi)$ implies $x \in q(\Phi \cap \hat{x})$,

where $\hat{x}$ denotes the principal ultrafilter containing $\{x\}$; $\Phi$ and $\Psi$ are in $F(X)$. Then the pair $(X, q)$ is called a convergence space. If $x \in q(\Phi)$, then we say that $\Phi$ $q$-converges to $x$. The filter $V_q(x)$ obtained by intersecting all filters which $q$-converge to $x$ is called the $q$-neighborhood filter at $x$. If $V_q(x)$ $q$-converges to $x$ for each $x \in X$, then $q$ is said to be pretopological and the pair $(X, q)$ is called a pretopological convergence space.

Let $C(X)$ be the set of all convergence structures on $X$, partially ordered as follows:

$q_1 \leq q_2$ iff $q_2(\Phi) \subseteq q_1(\Phi)$ for all $\Phi \in F(X)$.

If $q_1 \leq q_2$, then we say that $q_1$ is coarser than $q_2$, and $q_2$ is finer than $q_1$. By [5], we know that if $q_1$ is pretopological, then

$q_1 \leq q_2$ iff $V_{q_1}(x) \subseteq V_{q_2}(x)$ for all $x \in X$.

For any $q \in C(X)$, we define a related convergence structure $\pi(q)$, as follows:

$x \in \pi(q)(\Phi)$ iff $V_q(x) \subseteq \Phi$.

In this case, $\pi(q)$ is called the pretopological modification of $q$, and the pairs $(X, \pi(q))$ is called the pretopological modification of $(X, q)$. 
The n-th pretopological modification of convergence spaces

**Proposition 1** ([4]). \( \pi(q) \) is the finest pretopological convergence structure coarser than \( q \).

Let \( f \) be a map from \( X \) into \( Y \) and \( \Phi \) a filter on \( X \). Then \( f(\Phi) \) means the filter generated by \( \{ f(F) \mid F \in \Phi \} \). ([1])

**Proposition 2.** Let \( f : X \to Y \) be a map and \( \{ \Phi_i \mid i \in I \} \) a family of filters on \( F(X) \). Then \( f(\bigcap_{i \in I} \Phi_i) = \bigcap_{i \in I} f(\Phi_i) \).

**Proof.** Let \( B \in f(\bigcap_{i \in I} \Phi_i) \). Then there exists \( A \in \bigcap_{i \in I} \Phi_i \) such that \( f(A) \subseteq B \). Thus \( A \in \Phi_i \) and so \( f(A) \in f(\Phi_i) \) for all \( i \in I \). Finally, \( f(A) \in \bigcap_{i \in I} f(\Phi_i) \) and so \( B \in \bigcap_{i \in I} f(\Phi_i) \).

Conversely, let \( B \in \bigcap_{i \in I} f(\Phi_i) \). Then, for each \( i \in I \), there exists \( F \in \Phi_i \) such that \( f(F) \subseteq B \). Since \( F \subseteq f^{-1}(B) \), we obtain \( f^{-1}(B) \in \Phi_i \) for each \( i \in I \) and so \( f^{-1}(B) \subseteq \bigcap_{i \in I} \Phi_i \). While, since \( B \supseteq f(f^{-1}(B)) \subseteq f(\bigcap_{i \in I} \Phi_i) \), we obtain \( B \in f(\bigcap_{i \in I} \Phi_i) \). This completes the proof.

Let \( f \) be a map from a convergence space \( (X, q) \) to a convergence space \( (Y, p) \). Then \( f \) is said to be *continuous* at a point \( x \in X \), if the filter \( f(\Phi) \) on \( Y \) \( p \)-converges to \( f(x) \) for every filter \( \Phi \) on \( X \) \( q \)-converging to \( x \). If \( f \) is continuous at every point \( x \in X \), then \( f \) is said to be continuous.

Let \( q \) and \( q' \) be in \( C(X) \), and \( p \) and \( p' \) in \( C(Y) \). Then, we know that if \( q \leq q' \), \( p \geq p' \) and \( f : (X, q) \to (Y, p) \) is continuous, then \( f : (X, q') \to (Y, p') \) is continuous.

**Proposition 3** ([6]). (1) If \( f : (X, q) \to (Y, p) \) is continuous at \( x \in X \), then \( V_p(f(x)) \subseteq f(V_q(x)) \).

(2) If \( p \) is pretopological and \( V_p(f(x)) \subseteq f(V_q(x)) \), then \( f : (X, q) \to (Y, p) \) is continuous at \( x \in X \).

Let \( (X, q) \) be a convergence space. Then the set function \( I_q : P(X) \to P(X) \) is defined by as follows:

\[
I_q(A) = \{ x \in A \mid A \in V_q(x) \}
\]
for each $A \subset X$. Then, $I_q$ has the following properties:

(1) $I_q(\emptyset) = \emptyset$, $I_q(A) \subset A$

(2) $I_q(X) = X$

(3) $I_q(A \cap B) = I_q(A) \cap I_q(B)$

(4) $A \subset B$ implies $I_q(A) \subset I_q(B)$

for each $A, B \subset X$. But, in general, $I_q(I_q(A)) \neq I_q(A)$.

Also, we define a set function $I_q^n : P(X) \rightarrow P(X)$ for each $n \in N \cup \{\infty\}$, where $N$ is the set of positive integers, as follows:

$I_q^1(A) = I_q(A)$,

$I_q^{n+1}(A) = I_q(I_q^n(A))$ if $n \in N$,

$I_q^\infty(A) = \cap \{I_q^n(A) \mid n \in N\}$.

It is clear that $I_q^n(A \cap B) = I_q^n(A) \cap I_q^n(B)$ for each $n \in N \cup \{\infty\}$ and $A, B \subset X$.

Indeed, $I_q^n$ has all of the properties of a topological interior operator except idempotency.

Let $V_q^n(x) = \{A \subset X \mid x \in I_q^n(A)\}$. Then $V_q^n(x)$ is a filter on $X$ for each $n \in N \cup \{\infty\}$, and we know that for each $n \in N$,

$I_q^n(A) \supset I_q^{n+1}(A) \supset I_q^\infty(A)$ for each $A \subset X$,

and

$V_q^n(x) \supset V_q^{n+1}(x) \supset V_q^\infty(x)$ for each $x \in X$.

Define a structure $\pi_n(q)$ for each $n \in N \cup \{\infty\}$ as follows:

$x \in \pi_n(q)(\Phi)$ iff $V_q^n(x) \subset \Phi$

for each $\Phi \in F(X)$. It is not difficult to show that for each $n \in N \cup \{\infty\}$,

$V_{\pi_n(q)}(x) = V_q^n(x)$ for each $x \in X$,

$I_{\pi_n(q)}(A) = I_q^n(A)$ for all $A \subset X$.
and for each \( n \in \mathbb{N} \),

\[
q \geq \pi_n(q) \geq \pi_{n+1}(q) \geq \pi_\infty(q).
\]

While, since \( V_q(x) \subset \hat{x} \), we obtain \( x \in \pi_n(q)(\hat{x}) \) for each \( x \in X \). Also \( \Phi \subset \Psi \in \mathcal{F}(X) \) implies \( \pi_n(q)(\Phi) \subset \pi_n(q)(\Psi) \).

Let \( x \in \pi_n(q)(\Phi) \). Then \( V_q^n(x) \subset \Phi \). Since \( V_q^n(x) \subset \hat{x} \), we obtain \( V_q^n(x) \subset \Phi \cap \hat{x} \) and so \( x \in \pi_n(q)(\Phi \cap \hat{x}) \). Also, \( x \in \pi_n(q)(V_q^n(x)) = \pi_n(q)(V_{\pi_n(q)}(x)) \). Thus, \( \pi_n(q) \) is a pretopological convergence structure on \( X \), which is called the \( n \)-th pretopological modification of \( q \). Also, \( (X, \pi_n(q)) \) is called the \( n \)-th pretopological modification of \( (X, q) \).

**Proposition 4.** For \( q \in C(X) \), \( \cap \{ V_q^n(x) \mid n \in \mathbb{N} \} = V_q^\infty(x) \).

**Proof.** Let \( A \in V_q^\infty(x) \). Then \( x \in I_q^\infty(A) \) and so \( x \in I_q^n(A) \) for all \( n \in \mathbb{N} \). Thus, \( A \in V_q^n(x) \) for all \( n \in \mathbb{N} \).

Conversely, let \( A \in V_q^n(x) \) for all \( n \in \mathbb{N} \). Then \( A \in V_q^\infty(x) \). This completes the proof.

**Proposition 5.** Let \( f : (X, q) \to (Y, p) \) be a map and \( n \in \mathbb{N} \cup \{ \infty \} \). Then the following are equivalent:

(a) \( V_p^n(f(x)) = f(V_q^n(x)) \) for each \( x \in X \).

(b) \( f^{-1}(I_p^n(B)) = I_q^n(f^{-1}(B)) \) for each \( B \subset Y \).

**Proof.** First, assume that (a) is true, and let \( x \in f^{-1}(I_p^n(B)) \). Then \( f(x) \in I_p^n(B) \) and so \( B \in V_p^n(f(x)) = f(V_q^n(x)) \). Thus, \( f^{-1}(B) \in V_q^n(x) \) and so \( x \in I_q^n(f^{-1}(B)) \). Finally, \( f^{-1}(I_p^n(B)) \subset I_q^n(f^{-1}(B)) \). The reverse inequality is proved by the counter-order.

Next, assume that (b) is true, and let \( B \in V_p^n(f(x)) \). Then \( f(x) \in I_p^n(B) \) and so \( x \in f^{-1}(I_p^n(B)) = I_q^n(f^{-1}(B)) \). Thus \( f^{-1}(B) \in V_q^n(x) \) and so \( B \in f(V_q^n(x)) \). Finally, \( V_p^n(f(x)) \subset f(V_q^n(x)) \). The reverse inequality is proved by the counter-order. This completes the proof.

Let \( (X, q) \) be a convergence space, \( Y \) a nonempty set, and a map \( f : (X, q) \to Y \) a surjection. The convergence quotient structure \( p \) on \( Y \) is defined by specifying that for any \( y \in Y \) and \( \Psi \in \mathcal{F}(Y) \),

\[
y \in p(\Psi) \text{ iff there exist } x \in f^{-1}(y) \text{ and } \Phi \in \mathcal{F}(X) \text{ such that } \Psi \supset f(\Phi) \text{ and } x \in q(\Phi).
\]
In this case, \( f:(X,q) \rightarrow (Y,p) \) is called a *convergence quotient map* and the pair \((Y,p)\) is called a *convergence quotient space*.

Kent [6] proved that for a surjection \( f:(X,q) \rightarrow (Y,p) \), \( f \) is a convergence quotient map if and only if \( p \) is the finest convergence structure on \( Y \) relative to which \( f \) is continuous.

**Proposition 6 ([6]).** If \( f:(X,q) \rightarrow (Y,p) \) is a convergence quotient map, then, for each \( y \in Y \), \( V_p(y) = \cap \{ f(V_q(x)) \mid x \in f^{-1}(y) \} \).

3. Main Results

**Theorem 7.** Let \( f:(X,q) \rightarrow (Y,p) \) be a map. Then the following are equivalent:

(a) \( V_p(f(x)) = f(V_q(x)) \).

(b) \( V_p^n(f(x)) = f(V_q^n(x)) \) for each \( n \in \mathbb{N} \cup \{ \infty \} \).

**Proof.** It is clear that (b) implies (a). We will use the induction to prove that (a) implies (b). Assume that \( V_p^k(f(x)) = f(V_q^k(x)) \), and let \( B \in V_p^{k+1}(f(x)) \). Then \( f(x) \in I_p^{k+1}(B) = I_p(I_p^k(B)) \) and so \( I_p^k(B) \in V_p(f(x)) = f(V_q(x)) \). By the assumption and Proposition 5, \( f^{-1}(I_p^k(B)) = I_q^k(f^{-1}(B)) \subseteq V_q(x) \). Thus \( x \in I_q(I_q^k(f^{-1}(B))) = I_q^{k+1}(f^{-1}(B)) \) and so \( f^{-1}(B) \in V_q^{k+1}(x) \). Finally, \( B \in f(V_q^{k+1}(x)) \). This means \( V_p^{k+1}(f(x)) \subseteq f(V_q^{k+1}(x)) \). The reverse inequality is proved by the counter-order.

In that case \( n = \infty \), let \( B \in V_p^\infty(f(x)) \). Then \( f(x) \in I_p^\infty(B) \) and so \( f(x) \in I_p^n(B) \) for each \( n \in \mathbb{N} \). Thus \( B \in V_p^n(f(x)) = f(V_q^n(x)) \) for each \( n \in \mathbb{N} \). By Proposition 2, \( B \in \cap \{ f(V_q^n(x)) \mid n \in \mathbb{N} \} = f(\cap \{ V_q^n(x) \mid n \in \mathbb{N} \}) = f(V_q^\infty(x)) \). Finally, \( V_p^\infty(f(x)) \subseteq f(V_q^\infty(x)) \). The reverse inequality is proved by the counter-order. This completes the proof.

**Corollary 8.** If \( f:(X,q) \rightarrow (Y,p) \) is continuous, then for each \( n \in \mathbb{N} \cup \{ \infty \} \), \( f:(X,\pi_n(q)) \rightarrow (Y,\pi_n(p)) \) is continuous.

**Proof.** It is clear that “=” is replaced by “C” in the above Proposition 5 and Theorem 7, the statements are true. Consider that \( \pi_n(q) \) is pretopological for each \( n \in \mathbb{N} \cup \{ \infty \} \). Since \( V_{\pi_n(p)}(f(x)) = V_p^n(f(x)) \) and \( V_{\pi_n(q)}(x) = V_q^n(x) \), by Proposition 3, the proof is complete.
**Theorem 9.** Let \( f : (X, q) \to (Y, p) \) be continuous. Then the following hold:

1. If \( q \) is pretopological and for each \( y \in Y \) there exists \( x \in f^{-1}(y) \) such that \( V_p(y) = f(V_q(x)) \), then \( p \) is pretopological and \( f : (X, q) \to (Y, p) \) is a convergence quotient map.

2. If \( p \) is pretopological and \( f : (X, q) \to (Y, p) \) is a convergence quotient map, then for each \( y \in Y \) there exists \( x \in f^{-1}(y) \) such that \( V_p(y) = f(V_q(x)) \).

**Proof.** (1) Suppose that for each \( y \in Y \), there exists \( x \in f^{-1}(y) \) such that \( V_p(y) = f(V_q(x)) \). Since \( q \) is pretopological, we obtain \( x \in q(V_q(x)) \). From the continuity of \( f : (X, q) \to (Y, p) \), we obtain that \( y = f(x) \in p(V_q(x)) \) and so \( p \) is pretopological.

Let \( f : (X, q) \to (Y, r) \) be a convergence quotient map. Then \( p \leq r \). While, let \( \Psi \in F(Y) \) and \( y \in p(\Psi) \). Then \( \Psi \supset V_p(y) = f(V_q(x)) \) for some \( x \in f^{-1}(y) \). Since \( x \in q(V_q(x)) \) and \( f : (X, q) \to (Y, r) \) is a convergence quotient map, we obtain \( y \in r(\Psi) \). Thus \( p(\Psi) \subset r(\Psi) \) and so \( p \geq r \). Finally, \( p = r \). The proof is complete.

(2) Let \( y \in Y \). Since \( p \) is pretopological, we obtain \( y \in p(V_p(y)) \). Since \( f : (X, q) \to (Y, p) \) is a convergence quotient map, there exist \( x \in f^{-1}(y) \) and \( \Phi \in F(X) \) such that \( V_p(y) \supset f(\Phi) \) and \( x \in q(\Phi) \). Thus, \( V_q(x) \subset \Phi \) and so \( V_p(y) \supset f(V_q(x)) \). Since \( f : (X, q) \to (Y, p) \) is continuous, we obtain \( V_p(y) \subset f(V_q(x)) \). Finally, \( V_p(y) = f(V_q(x)) \). This completes the proof.

**Theorem 10.** If \( p \) is pretopological and \( f : (X, q) \to (Y, p) \) is a convergence quotient map, then the following hold for each \( n \in N \cup \{ \infty \} \):

1. For each \( y \in Y \), there exists \( x \in f^{-1}(y) \) such that \( V_{p^n}(y) = f(V_{q^n}(x)) \).

2. \( f : (X, \pi_n(q)) \to (Y, \pi_n(p)) \) is a convergence quotient map.

3. For each \( y \in Y \), \( V_{p^n}(y) = f(\cap \{ V_{q^n}(x) \mid x \in f^{-1}(y) \}) \).

**Proof.** (1) By Corollary 8, \( f : (X, \pi_n(q)) \to (Y, \pi_n(p)) \) is continuous. Since \( f : (X, q) \to (Y, p) \) is a convergence quotient map and \( p \) is pretopological, by Theorem 9 (2), for each \( y \in Y \), there exists \( x \in f^{-1}(y) \) such that \( V_p(y) = f(V_q(x)) \). Thus, by Theorem 7, \( V_{p^n}(y) = f(V_{q^n}(x)) \) for each \( n \in N \cup \{ \infty \} \).
(2) Since $V_{\pi_n(p)}(y) = f(V_{\pi_n(q)}(x))$ and $\pi_n(q)$ is pretopological, by Theorem 9 (1), $f:(X, \pi_n(q)) \to (Y, \pi_n(p))$ is a convergence quotient map.

(3) By the above (2) and Proposition 6, the proof is complete.

References


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